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Solutions of Direct Geodetic Problem in Navigational Applications

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ABSTRACT: Solutions of such navigational problems as positions from ranges, bearings and courses without any simplifications for a plane or a sphere, by an application of solutions of direct geodetic problem are presented. The rigorous, rapid, non-iterative solution of the direct geodetic problem according to Sodano, for any length of geodesics, is attached.

1 INTRODUCTION

The navigational problem is as follows (Fig. 1):

- we have known geographic coordinates of position $P' - \varphi_{P'}, \lambda_{P'}$,
- we have ranges, bearings or courses from position P' to position P,
- we search for geographic coordinates of position $P - \varphi_P, \lambda_P$,

The most common solution of such a navigational problem is a rather strange combination of flat and ellipsoidal calculations:

- conversion ranges, bearings and courses, by solving flat triangles, to δx, δy increments in a flat rectangular coordinate frame (with the y-axis pointing north),
- conversion rectangular δx, δy flat increments to geographic coordinates increments $\delta\varphi$, $\delta\lambda$, on the reference ellipsoid, by the equations

$$\delta\varphi = \frac{\delta y}{R_M(\varphi_{P'})} \tag{1}$$

$$\delta\lambda = \frac{\delta x}{R_N(\varphi_{P'})\cos(\varphi_{P'})} \tag{2}$$

where $R_M(\varphi_{P'})$ = the radius of curvature in meridian for P'; and $R_N(\varphi_{P'})$ = the radius of curvature in the prime vertical for P'

given by the equations

$$R_M = \frac{a_0(1-e^2)}{\sqrt{(1-e^2\sin^2\varphi_{P'})^3}} \tag{3}$$

$$R_N = \frac{a_0}{\sqrt{1-e^2\sin^2\varphi_{P'}}} \tag{4}$$

where a_0 = the semi-major axis of the reference ellipsoid; and e = eccentricity

and finally

$$\varphi_P = \varphi_{P'} + \delta\varphi \tag{5}$$

$$\lambda_P = \lambda_{P'} + \delta\lambda \tag{6}$$

if east longitude and north latitude are considered positive and west longitude and south latitude are considered negative.

Apart from the obvious errors of assuming the part of the ellipsoid a plane there are also errors hidden in Equations 1 and 2 – although Equations 3 and 4 are accurate for the ellipsoid – the errors of assuming the main radii of curvature constant at points P' and P.

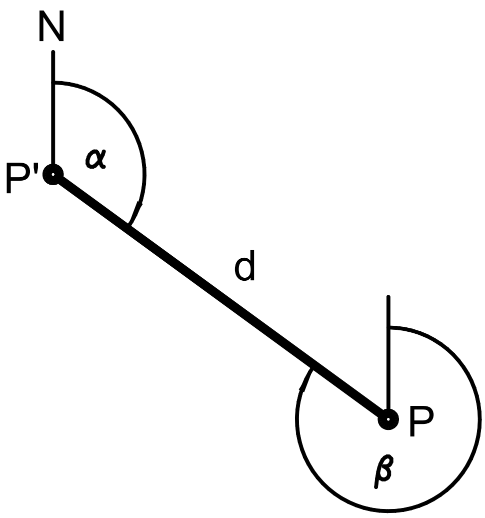

Figure 1.Definition of the problem and the position from range and bearings

2 ERROR OF ASSUMING THE PART OF ELLIPSOID TO BE A PLANE

The spherical excess in an equilateral spherical triangle with sides d and spherical radius R is

$$\varepsilon = \frac{d^2\sqrt{3}}{4R^2} \tag{7}$$

For d = 22 km (≈ 12 n.m.) and R = 6370 km, we get

$$\varepsilon \approx 5\times10^{-6}\ \text{rad}$$

This gives linear changes in the range of 11 cm.

3 ERRORS OF ASSUMING THE MAIN RADII CONSTANT AT POINTS P′ AND P

A better approximation of Equation 1 should be (Lenart 1985)

$$\delta\varphi^* = \frac{\delta y}{R_M\left(\frac{\varphi_{P'}+\varphi_P}{2}\right)} = \frac{\delta y}{R_M(\varphi_{P'}) + \frac{1}{2}\delta\varphi\frac{dR_M}{d\varphi}} \tag{8}$$

Defining the resulting error as

$$\Delta\delta\varphi = \delta\varphi - \delta\varphi^* \tag{9}$$

we get

$$\Delta\delta\varphi = \delta y \frac{\frac{1}{2}\delta\varphi\frac{dR_M}{d\varphi}}{R_M(\varphi_{P'})\left(R_M(\varphi_{P'}) + \frac{1}{2}\delta\varphi\frac{dR_M}{d\varphi}\right)} \tag{10}$$

which with

$$\delta y = \delta\varphi\left(R_M(\varphi_{P'}) + \frac{1}{2}\delta\varphi\frac{dR_M}{d\varphi}\right) \tag{11}$$

yields

$$\Delta\delta\varphi = \frac{\frac{1}{2}(\delta\varphi)^2\frac{dR_M}{d\varphi}}{R_M(\varphi_{P'})} \tag{12}$$

After substitution

$$\frac{dR_M}{d\varphi} \approx \frac{3}{2}e^2\sin 2\varphi R_M \tag{13}$$

we finally get

$$\Delta\delta\varphi \approx \frac{3}{4}e^2(\delta\varphi)^2\sin 2\varphi \tag{14}$$

where, if δφ is in radians the result is also in radians.

For example, if

$$\delta\varphi = 12'' \approx 35\times10^{-4}\,\text{rad}$$

$$\varphi = 45°$$

we get

$$\Delta\delta\varphi \approx \frac{3}{4}\times\frac{1}{150}\times1225\times10^{-8}\ \text{rad} \approx 6.125\times10^{-8}\ \text{rad}$$

$$\approx .0126'' \approx 39\,\text{cm}$$

In the case of error in longitude we have, in accordance with Equation 2,

$$\delta\lambda^* = \frac{\delta x}{\left(R_N(\varphi_{P'}) + \frac{1}{2}\delta\varphi\frac{dR_N}{d\varphi}\right)\cos\left(\varphi_{P'} + \frac{\delta\varphi}{2}\right)} \tag{15}$$

Then

$$\Delta\delta\lambda = \delta\lambda - \delta\lambda^* \tag{16}$$

and, after substitution

$$\cos\left(\varphi_{P'} + \frac{\delta\varphi}{2}\right) \approx \cos\varphi_{P'} - \frac{\delta\varphi}{2}\sin\varphi_{P'} \tag{17}$$

$$\delta x = \delta\lambda\left(R_N(\varphi_{P'}) + \frac{1}{2}\delta\varphi\frac{dR_N}{d\varphi}\right)\cos\left(\varphi_{P'} + \frac{\delta\varphi}{2}\right) \tag{18}$$

After simplification

$$\Delta\delta\lambda \approx \frac{\frac{1}{2}\delta\varphi\delta\lambda\left(\frac{dR_N}{d\varphi}\cos\varphi_{P'} - R_N\sin\varphi_{P'}\right)}{R_N\cos\varphi_{P'}} \tag{19}$$

Since

$$\frac{dR_N}{d\varphi} \approx \frac{1}{2}e^2\sin 2\varphi R_N \tag{20}$$

then finally

$$\Delta\delta\lambda \approx -\frac{1}{2}\delta\varphi\delta\lambda\tan\varphi \quad (21)$$

If $\delta\varphi$ and $\delta\lambda$ are in radians, the result is also in radians.

For example, if

$\varphi = 80°$

$\delta\varphi = 12' \approx 35\times10^{-4}\,\mathrm{rad}$

$\delta\lambda = 69' \approx 201\times10^{-4}\,\mathrm{rad}$ (which relates to 12′ on the equator)

then

$$\Delta\delta\lambda \approx -\frac{1}{2}\times35\times201\times10^{-8}\times5.7\ \mathrm{rad} \approx -41.36''$$

At that latitude, this corresponds to –222 m!

4 THE DIRECT GEODETIC PROBLEM

It can be seen from the above, that the errors of simplifications are neglectable or significant, depending on the required accuracy and the values of φ, $\delta\varphi$ and $\delta\lambda$, but all of them are systematic and are integrated in dead reckoning.

These simplifications have been necessary to reduce the number of calculations on the ellipsoid and justified in times of manual mechanical or electronic calculators, but are completely unnecessary and unjustified in times of computer calculations. Therefore we will directly apply the solution of the problem known in geodesy as direct geodetic problem.

In the solution of the direct geodetic problem (Fig. 2) from the given coordinates φ_1, λ_1 and azimuth $\alpha_{1\text{-}2}$ at the start of geodesics P_1 and their length S are calculated coordinates φ_2, λ_2 of the endpoint P_2 and the reversed azimuth $\alpha_{2\text{-}1}$, on any reference ellipsoid.

E. M. Sodano (Sodano 1958, 1965, 1967) from Helmert's classical iterative formulae derived a rigorous non-iterative procedure, for any length of geodesics and for any required accuracy, which is attached in Appendix A. This procedure will be used in this paper in the formal form

$$\varphi_2, \lambda_2 = \mathrm{SDGP}\,(\varphi_1, \lambda_1, \alpha_{1\text{-}2}, S) \quad (22)$$

$$\alpha_{2\text{-}1} = \mathrm{SDGP}\,(\varphi_1, \lambda_1, \alpha_{1\text{-}2}, S) \quad (23)$$

5 APPLICATION OF THE DIRECT GEODETIC PROBLEM

5.1 *Position from range and bearings*

We search for the position $P(\varphi_P, \lambda_P)$ for which we have the range d and the bearing α from, or the bearing β to, known position $P'(\varphi_{P'}, \lambda_{P'})$ (Fig. 1).

The solution is

$$\varphi_P, \lambda_P = \mathrm{SDGP}\,(\varphi_{P'}, \lambda_{P'}, \alpha, d) \quad (24)$$

or

$$\varphi_P, \lambda_P = \mathrm{SDGP}\,(\varphi_{P'}, \lambda_{P'}, \beta - 180°, d) \quad (25)$$

5.2 *Dead reckoned position*

We search for the position $P(\varphi_P, \lambda_P)$ dead reckoned from known position $P'(\varphi_{P'}, \lambda_{P'})$ with the speed over ground V_g and the course over ground C_g during the time interval Δt (Fig. 3).

The solution is

$$\varphi_P, \lambda_P = \mathrm{SDGP}\,(\varphi_{P'}, \lambda_{P'}, C_g, V_g\,\Delta t) \quad (26)$$

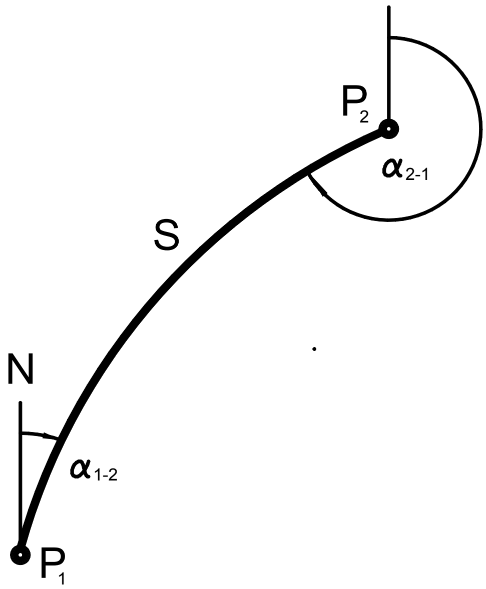

Figure 2. Direct and inverse geodetic problem

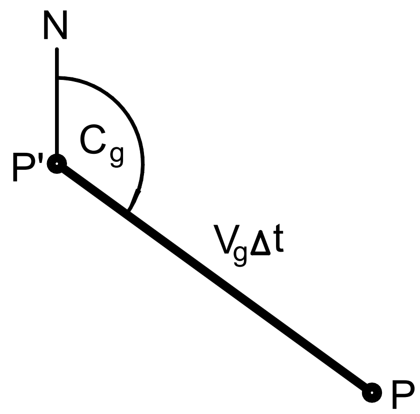

Figure 3. Dead reckoned position

5.3 *Position from two ranges*

We search for the position $P(\varphi_P, \lambda_P)$ for which we have two ranges d_1 and d_2 to known positions $P'_1(\varphi_{P'1}, \lambda_{P'1})$ and $P'_2(\varphi_{P'2}, \lambda_{P'2})$ (Fig. 4).

The solution is iterative

$$\varphi_{P1}, \lambda_{P1} = \text{SDGP}(\varphi_{P'1}, \lambda_{P'1}, \alpha_1 = \text{var}, d_1) \quad (27)$$

$$\varphi_{P2}, \lambda_{P2} = \text{SDGP}(\varphi_{P'2}, \lambda_{P'2}, \alpha_2 = \text{var}, d_2) \quad (28)$$

where α_1 and α_2 are adjusted by any small increments until e.g.

$$(\varphi_{P2} - \varphi_{P1})^2 + (\lambda_{P2} - \lambda_{P1})^2 \cos\varphi_{P2} \cos\varphi_{P1} = \min \quad (29)$$

This iterative process, although looks as very complicated, is very fast and simple with using e.g. the Solver in Microsoft Excel.

5.4 *Position from two bearings*

We search for the position $P(\varphi_P, \lambda_P)$ for which we have the bearing α_1 from, or the bearing β_1 to, known position $P'_1(\varphi_{P'1}, \lambda_{P'1})$ and the bearing α_2 from, or the bearing β_2 to, known positions $P'_2(\varphi_{P'2}, \lambda_{P'2})$ (Fig. 4).

The solution is iterative

$$\varphi_{P1}, \lambda_{P1} = \text{SDGP}(\varphi_{P'1}, \lambda_{P'1}, \alpha_1, d_1 = \text{var}) \quad (30)$$

$$\varphi_{P2}, \lambda_{P2} = \text{SDGP}(\varphi_{P'2}, \lambda_{P'2}, \alpha_2, d_2 = \text{var}) \quad (31)$$

or

$$\varphi_{P1}, \lambda_{P1} = \text{SDGP}(\varphi_{P'1}, \lambda_{P'1}, \beta - 180°, d_1 = \text{var}) \quad (32)$$

$$\varphi_{P2}, \lambda_{P2} = \text{SDGP}(\varphi_{P'2}, \lambda_{P'2}, \beta - 180°, d_2 = \text{var}) \quad (33)$$

or any combination of the above, where d_1 and d_2 are adjusted by any small increments until e.g. Equation 29 is fulfilled.

5.5 *Position from range and bearing to different positions*

We search for position $P(\varphi_P, \lambda_P)$ for which we have the range d_1 to known position $P'_1(\varphi_{P'1}, \lambda_{P'1})$ and the bearing α_2 from, or the bearing β_2 to, known positions $P'_2(\varphi_{P'2}, \lambda_{P'2})$ (Fig. 4).

The solution is iterative

$$\varphi_{P1}, \lambda_{P1} = \text{SDGP}(\varphi_{P'1}, \lambda_{P'1}, \alpha_1 = \text{var}, d_1) \quad (34)$$

$$\varphi_{P2}, \lambda_{P2} = \text{SDGP}(\varphi_{P'2}, \lambda_{P'2}, \alpha_2, d_2 = \text{var}) \quad (35)$$

or

$$\varphi_{P2}, \lambda_{P2} = \text{SDGP}(\varphi_{P'2}, \lambda_{P'2}, \beta - 180°, d_2 = \text{var}) \quad (36)$$

where α_1 and d_2 are adjusted by any small increments until e.g. Equation 29 is fulfilled.

5.6 *Position from any combination of ranges and bearings*

The above can be easily extended to any number of combination of ranges and bearings - we search for position $P(\varphi_P, \lambda_P)$ for which we have n ranges d or bearings α from, or bearings β to, n known positions $P'(\varphi_{P'}, \lambda_{P'})$ (Fig. 5).

The solution is iterative

$$\varphi_{P1}, \lambda_{P1} = \text{SDGP}(\varphi_{P'1}, \lambda_{P'1}, \alpha_1 = \text{var}, d_1) \quad (37)$$

$$\varphi_{P2}, \lambda_{P2} = \text{SDGP}(\varphi_{P'2}, \lambda_{P'2}, \alpha_2, d_2 = \text{var}) \quad (38)$$

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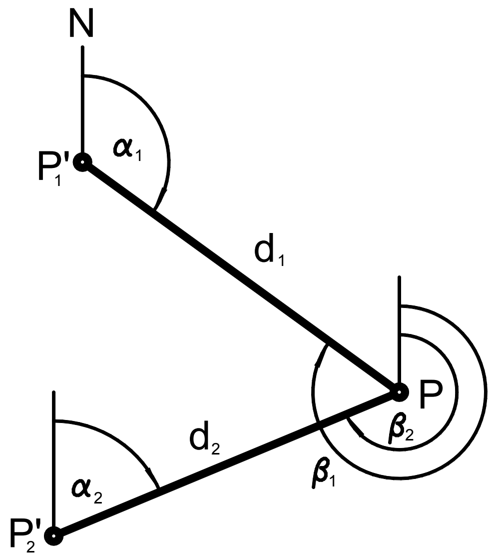

Figure 4. Positions from two ranges or bearings

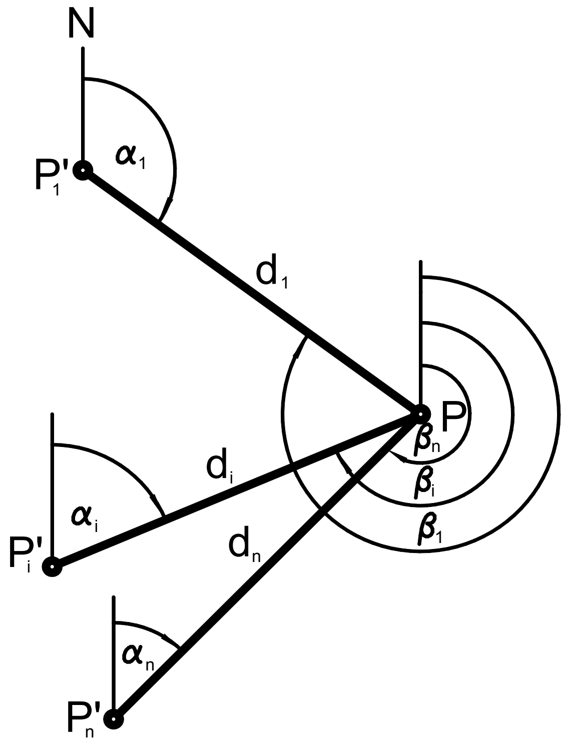

Figure 5. Position from n ranges and bearings

$\varphi_{Pi}, \lambda_{Pi} = \text{SDGP}\ (\varphi_{P'i}, \lambda_{P'i}, \beta_i - 180°, d_i = \text{var})$ (39)

……………………………………….

$\varphi_{Pn}, \lambda_{Pn} = \text{SDGP}\ (\varphi_{P'n}, \lambda_{P'n}, \beta_n - 180°, d_n = \text{var})$ (40)

where d or α or β are adjusted by any small increments until e.g.

$$\sum_{i=1}^{n}(\varphi_{Pi}-\bar{\varphi})^2+(\lambda_{Pi}-\bar{\lambda})^2\cos^2\varphi_{Pi}=\min \tag{41}$$

where

$$\bar{\varphi}=\frac{1}{n}\sum_{i=1}^{n}\varphi_{Pi} \tag{42}$$

$$\bar{\lambda}=\frac{1}{n}\sum_{i=1}^{n}\lambda_{Pi} \tag{43}$$

It is worth mentioning, that we will achieve the least square errors position in the case of excessive number of position lines.

5.7 *Position lines of different accuracies*

In the case of position lines of different accuracies we can extend Equation 29 or 41 with weights – e.g. reciprocal of mean square errors m_i

$$\sum_{i=1}^{n}\frac{(\varphi_{Pi}-\bar{\varphi})^2+(\lambda_{Pi}-\bar{\lambda})^2\cos^2\varphi_{Pi}}{m_i^2}=\min \tag{44}$$

5.8 *Bearings for long ranges*

For long ranges

$$\alpha \neq \beta - 180° \tag{45}$$

If this difference is significant, for Section 5.1, we at first iteratively search for α from the equation

$\alpha_{2\text{-}1} = \text{SDGP}\ (\varphi_{P'}, \lambda_{P'}, \alpha = \text{var}, d)$ (46)

until

$\alpha_{2\text{-}1} = \beta$ (47)

and then

$\varphi_P, \lambda_P = \text{SDGP}\ (\varphi_{P'}, \lambda_{P'}, \alpha, d)$ (48)

For Section 5.4 and 5.5 Equations 32, 33, 36, 39 and 40 becomes respectively to

$\varphi_{Pi}, \lambda_{Pi} = \text{SDGP}\ (\varphi_{P'i}, \lambda_{P'i}, \alpha_i = \text{var}, d_i = \text{var})$ (49)

and additionally

$\alpha_{2\text{-}1i} = \text{SDGP}\ (\varphi_{P'i}, \lambda_{P'i}, \alpha_i = \text{var}, d_i = \text{var})$ (50)

as well as Equations 29 and 41 should be supplemented by the component e.g.

$$(\beta_i-\alpha_{2-1i})^2\left[\left(\frac{d_i\sin\beta_i}{R_N(\varphi_{Pi})\cos\varphi_{Pi}}\right)^2+\left(\frac{d_i\cos\beta_i}{R_M(\varphi_{Pi})}\right)^2\right] \tag{51}$$

for each β_i.

6 ACCURACY OF THE SOLUTION OF THE DIRECT GEODETIC PROBLEM

"The accuracy of geodetic distances computed through the e^2, e^4, e^6 order for very long geodesics is within a few meters, centimeters and tenth of millimeters respectively. Azimuths are good to tenth, thousandths and hundreds thousandths of a second. Further improvement of results occurs for shorter lines" (Sodano 1958).

7 DIRECT COMPUTATION FORM SIMPLIFIED

For shorter distances (the abovementioned "very long geodesics" means even 20 000 km) or lower required accuracies we can use equations from Appendix A reduced to e^2 and f order. Therefore Equation A 9 becomes to

$$\Phi_0=\Phi_S-\frac{1}{2}a_1e'^2\sin\Phi_S+\frac{1}{4}m_1e'^2(-\Phi_S+\sin\Phi_S\cos\Phi_S) \tag{52}$$

and Equation A 12 becomes to

$$L=-f\Phi_S\cos\beta_0+\gamma \tag{53}$$

8 CIRCULAR FUNCTIONS

The angles $\alpha_{2\text{-}1}$ and γ from Equations A 10 and A 11 have to be calculated with the circular function $\tan^{-1}()$, but this function gives solutions in the range (-90°, 90°). For full range (0°, 360°) retrieving tables of quadrants are used in Sodano 1965.

For computer calculations a special procedure should be used to retrieve the full range (0°, 360°) from the signs of the numerator N and the denominator D and to detect and support a division by zero case e.g.:

For

$$\text{angle}=\tan^{-1}\frac{N}{D}$$

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IF D ≠ 0 THEN ANGLE = ATN(N/D)
 IF D < 0 THEN ANGLE = ANGLE + 180°: END IF
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ELSE

ANGLE = (2 - SIGN(N))*ABS(SIGN(N))*90°

END IF

IF ANGLE<0 THEN ANGLE=ANGLE+360°: END IF

9 CONCLUSIONS

Presented procedures are quite general and universal. They can be used for any number of ranges and bearings, any combinations of ranges and bearings, any ranges – from meters up to 20 000 km, with almost any required accuracy, on any reference ellipsoid and can calculate the optimal position according to any objective function.

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APPENDIX A

Direct computation form (Sodano 1965)

Given: φ_1, λ_1, α_{1-2}, S

Required: φ_2, λ_2, α_{1-2}

Reference ellipsoid: a_0, b_0 = semi-major and semi-minor axes

Flattening

$$f = 1 - \frac{b_0}{a_0} \quad \text{(A 1)}$$

Second eccentricity squared

$$e'^2 = \frac{{a_0}^2 - {b_0}^2}{{b_0}^2} \quad \text{(A 2)}$$

$$\tan\beta_1 = (1-f)\tan\varphi_1 \quad \text{(A 3)}$$

$$\cos\beta_0 = \cos\beta_1 \sin\alpha_{1-2} \quad \text{(A 4)}$$

$$g = \cos\beta_1 \cos\alpha_{1-2} \quad \text{(A 5)}$$

$$\Phi_S = \frac{S}{b_0} \quad \text{(A 6)}$$

$$m_1 = (1 + \frac{e'^2}{2}\sin^2\beta_1)(1 - \cos^2\beta_0) \quad \text{(A 7)}$$

$$a_1 = (1 + \frac{e'^2}{2}\sin^2\beta_1)(\sin^2\beta_1 \cos\Phi_S + g\sin\beta_1 \sin\Phi_S) \quad \text{(A 8)}$$

$$\begin{aligned}\Phi_0 &= \Phi_S - \frac{1}{2}a_1 e'^2 \sin\Phi_S \\ &+ \frac{1}{4}m_1 e'^2(-\Phi_S + \sin\Phi_S\cos\Phi_S) \\ &+ \frac{5}{8}{a_1}^2 e'^4 \sin\Phi_S \cos\Phi_S \\ &+ {m_1}^2 e'^4(\frac{11}{64}\Phi_S - \frac{13}{64}\sin\Phi_S\cos\Phi_S \\ &- \frac{1}{8}\Phi_S\cos^2\Phi_S + \frac{5}{32}\sin\Phi_S\cos^3\Phi_S) \\ &+ a_1 m_1 e'^4(\frac{3}{8}\sin\Phi_S + \frac{1}{4}\Phi_S\cos\Phi_S \\ &- \frac{5}{8}\sin\Phi_S\cos^2\Phi_S)\end{aligned} \quad \text{(A 9)}$$

$$\tan\alpha_{2-1} = \frac{\cos\beta_0}{g\cos\Phi_0 - \sin\beta_1\sin\Phi_0} \quad \text{(A 10)}$$

$$\tan\gamma_1 = \frac{\sin\Phi_0 \sin\alpha_{1-2}}{\cos\beta_1\cos\Phi_0 - \sin\beta_1\sin\Phi_0\cos\alpha_{1-2}} \quad \text{(A 11)}$$

$$\begin{aligned}L &= [-f\Phi_S + \frac{3}{2}f^2 a_1 \sin\Phi_S \\ &+ \frac{3}{4}f^2 m_1(\Phi_S - \sin\Phi_S\cos\Phi_S)]\cos\beta_0 + \gamma_1\end{aligned} \quad \text{(A 12)}$$

$$\lambda_2 = \lambda_1 + L \quad \text{(A 13)}$$

$$\sin\beta_2 = \sin\beta_1\cos\Phi_0 + g\sin\Phi_0 \quad \text{(A 14)}$$

$$\tan\varphi_2 = \frac{\tan\beta_2}{(1-f)} \quad \text{(A 15)}$$