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So, What is Actually the Distance from the Equator to the Pole? – Overview of the Meridian Distance Approximations

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ABSTRACT: In the paper the author presents overview of the meridian distance approximations. He would like to find the answer for the question what is actually the distance from the equator to the pole - the polar distance. In spite of appearances this is not such a simple question. The problem of determining the polar distance is a great opportunity to demonstrate the multitude of possible solutions in common use. At the beginning of the paper the author discusses some approximations and a few exact expressions (infinite sums) to calculate perimeter and quadrant of an ellipse, he presents convenient measurement units of the distance on the surface of the Earth, existing methods for the solution of the great circle and great elliptic sailing, and in the end he analyses and compares geodetic formulas for the meridian arc length.

1 INTRODUCTION

Unfortunately, from the early days of the development of the basic navigational software built into satellite navigational receivers and later into electronic chart systems, it has been noted that for the sake of simplicity and a number of other, often incomprehensible reasons, this navigational software is often based on the simple methods of limited accuracy. It is surprising that even nowadays, at the beginning of the twenty-first century, the use of navigational software is still used in a loose manner, sometimes ignoring basic computational principles and adopting oversimplified assumptions and errors such as the wrong combination of spherical and ellipsoidal calculations (while in car navigation systems – even primitive simple calculations on flat surfaces) in different steps of the solution of a particular sailing problem. The lack of official standardization on both the "accuracy required" and the equivalent "methods employed", in conjunction to the "black box solutions" provided by GNSS navigational receivers and navigational systems (ECDIS and ECS [Weintrit, 2009]) suggest the necessity of a thorough examination, modification, verification and unification of the issue of sailing calculations for navigational systems and receivers. The problem of determining the distance from the equator to the pole is a great opportunity to demonstrate the multitude of possible solutions in common use.

2 THE MAIN QUESTION AND FIVE THE BEST AD HOC ANSWERS

Well, let's put the title question - what is actually distance from the Equator to the Pole? And let us consider what actually answer would we expect? There will answers simple, crude, naive, almost primitive, but also very sophisticated and refined, full of mathematics. As it might seem at first glance, surely the problem is not trivial.

2.1 *Answer No.1*

It's exactly 10,000 km. This is because the definition of a meter is 1 10,000,000th of the distance from the North Pole to the equator. So it's exactly 10,000,000 meters from the North Pole to the equator, which is exactly 10,000 km.

2.2 *Answer No.2*

10,002 kilometres. The original definition of a kilometre was 1/10,000 of the distance from the equator to the North Pole, but measurements have improved.

2.3 *Answer No.3*

Easy, there are 90 degrees of distance from the equator to the North Pole. Each degree has 60 minutes, each minute = 1 nautical mile, therefore 60 x 90 = 5,400 nautical miles.

2.4 *Answer No.4*

Angle between the equator and North Pole is 90°. 1 nautical mile = 1852 meters = 1′; 1° = 60′; just multiply 60 x 90 x 1852. The answer is 10,000,800 m.

2.5 *Answer No.5*

If the question is: what is the distance from the North Pole to the equator in degrees? - the answer is much easier.

The measure of a circle in degrees is 360 degrees. So the distance from Pole to equator is one quarter of this; namely, 90 degrees.

2.6 *What is Important in That Calculation?*

Frankly speaking, all five answers are correct, and also ... completely wrong. First of all we should decide what length unit we will use for the measurement, what model of the Earth will be used for our calculations, and the accuracy of the result we expect.

We know already that the Earth is not a sphere; therefore our calculations should be a bit more difficult. We will use the ellipsoid of revolution. Early literature uses the term oblate spheroid to describe a sphere "squashed at the poles". Modern literature uses the term "ellipsoid of revolution" although the qualifying words "of revolution" are usually dropped. An ellipsoid which is not an ellipsoid of revolution is called a tri-axial ellipsoid. Spheroid and ellipsoid are used interchangeably in this paper. Currently we use to navigate the ellipsoid WGS-84 (*World Geodetic System 1984*). The WGS-84 meridional ellipse has an ellipticity ε = 0.081819191.

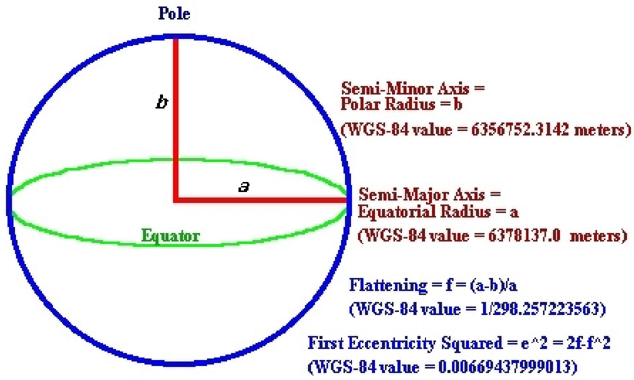

Figure 1. Parameters of the ellipsoid WGS-84 [Dana, 1994]

3 MEASUREMENT OF THE DISTANE ON SURFACE OF THE EARTH

We have to decide what unit of measurement we would like to use for measuring the distance: miles or metres. While the measure of one meter has been strictly defined, miles seem to be made of chewing gum. There are a lot of different miles, some of them are measures of a fixed length, such as: geographical mile, International Nautical Mile (INM), statue mile, other of variable length dependent on the latitude of location of measurement, such as: nautical mile or sea mile.

3.1 *Geographical Mile*

Distances on the surface of a sphere or an ellipsoid of revolution are expressed in a natural way in units of the length of one minute of arc, measured along the equator. This unit is known as the Geographical Mile. Its value is determined by the dimensions of the spheroid in use. We will use it throughout in our treatment of navigational methods. Its length varies according to the ellipsoid which is being used as the model but, in these units, the radius of the Earth is fixed at a value of $108{,}000/\pi$. The length of one minute of arc of the equator on the surface of the WGS-84 ellipsoid is approximately 1,855.3284 metres.

3.2 *The International Nautical Mile*

The international nautical mile was defined by the First International Extraordinary Hydrographic Conference, Monaco (1929) as exactly 1852 metres. This is the only definition in widespread current use, and is the one accepted by the International Hydrographic Organization (IHO) and by the International Bureau of Weights and Measures (BIPM). Before 1929, different countries had different definitions, and the United Kingdom, the United States, the Soviet Union and some other countries did not immediately accept the international value.

Both the Imperial and U.S. definitions of the nautical mile were based on the Clarke (1866) spheroid: they were different approximations to the length of one minute of arc along a great circle of a sphere having the same surface area as the Clarke spheroid. The United States nautical mile was defined as 1,853.248 metres (6,080.20 U.S. feet, based on the definition of the foot in the Mendenhall Order of 1893): it was abandoned in favour of the international nautical mile in 1954. The Imperial (UK) nautical mile, also known as the Admiralty mile, was defined in terms of the knot, such that one nautical mile was exactly 6,080 international feet (1,853.184 m): it was abandoned in 1970 and, for legal purposes, old references to the obsolete unit are now converted to 1,853 metres exactly [Weintrit, 2010].

3.3 *Nautical Mile*

A nautical mile is a unit of measurement used on water by sailors and/or navigators in shipping and aviation. It is the average length of one minute of one degree along a great circle of the Earth. One nautical mile corresponds to one minute of latitude. Thus, degrees of latitude are approximately 60 nautical miles apart. By contrast, the distance of nautical miles between degrees of longitude is not constant because lines of longitude become closer together as they converge at the poles.

Each country can keep different, arbitrarily selected value of the nautical mile, but most of them use the International Nautical Mile, although in the past it was different.

The unit used by the United Kingdom until 1970 was the British Standard nautical mile of 6,080 ft or 1,853.18 m.

Today, one nautical mile still equals exactly the internationally agreed upon measure of 1,852 meters (6,076 feet). One of the most important concepts in understanding the nautical mile though is its relation to latitude.

3.4 *The Sea Mile*

The sea mile is the length of 1 minute of arc, measured along the meridian, in the latitude of the position; its length varies both with the latitude and with the dimensions of the spheroid in use.

The sea mile is an ambiguous unit, with the following possible meanings:

In English usage, a sea mile is, for any latitude, the length of one minute of latitude at that latitude. It varies from about 1,842.9 metres (6,046 ft) at the equator to about 1,861.7 metres (6,108 ft) at the poles, with a mean value of 1,852.3 metres (6,077 ft). The international nautical mile was chosen as the integer number of metres closest to the mean sea mile.

American use has changed recently. The glossary in the 1966 edition of Bowditch defines a "sea mile" as a "nautical mile". In the 2002 edition [Bowditch, 2002], the glossary says: "An approximate mean value of the nautical mile equal to 6,080 feet; the length of a minute of arc along the meridian at latitude 48°."

The sea mile has also been defined as 6,000 feet or 1,000 fathoms, for example in Dresner's *Units of Measurement* [Dresner, 1971]. Dresner includes a remark to the effect that this must not be confused with the nautical mile. Richard Norwood in *The Seaman's Practice* (1637) determined that 1/60th of a degree of any great circle on Earth's surface was 6,120 feet (vs. the modern value of 6,080 feet). However, he stated: "if any man think it more safe and convenient in Sea-reckonings" he may assign 6,000 feet to a mile, relying on context to determine the type of mile.

3.5 *The Statue Mile*

The statue mile is the unit of distance of 1,760 yards or 5,280 ft) 1609.3 m. The difference between a mile and a statute mile is historical, rather than practical.

Hundreds of years a mile meant different things to different people. It became necessary, eventually, for a mile to be the same distance for all concerned. During the reign of Queen Elizabeth I, a statute was passed by the English Parliament that standardized the measurement of a mile, thus giving rise to the term 'statute' mile. The measurement of a mile at 5,280 feet is now accepted almost everywhere in the world.

3.6 *History of the Mile*

The nautical mile was historically defined as a minute of arc along a meridian of the Earth (North-South), making a meridian exactly 180×60 = 10,800 historical nautical miles. It can therefore be used for approximate measures on a meridian as change of latitude on a nautical chart. The originally intended definition of the metre as 10^{-7} of a half-meridian arc makes the mean historical nautical mile exactly $(2\times10^{7})/10,800$ = 1,851.851851… historical metres. Based on the current IUGG meridian of 20,003,931.4585 (standard) metres the mean historical nautical mile is 1,852.216 m.

The historical definition differs from the length-based standard in that a minute of arc, and hence a nautical mile, is not a constant length at the surface of the Earth but gradually lengthens in the north-south direction with increasing distance from the equator, as a corollary of the Earth's oblateness, hence the need for "mean" in the last sentence of the previous paragraph. This length equals about 1,861 metres at the poles and 1,843 metres at the Equator.

Other nations had different definitions of the nautical mile. This variety, in combination with the complexity of angular measure described above and the intrinsic uncertainty of geodetically derived units, mitigated against the extant definitions in favour of a simple unit of pure length. International agreement was achieved in 1929 when the IHB adopted a definition of one international nautical mile as being equal to 1,852 metres exactly, in excellent agreement (for an integer) with both the above-mentioned values of 1,851.851 historical metres and 1,852.216 standard metres.

The use of an angle-based length was first suggested by Edmund Gunter (of Gunter's chain fame). During the 18th century, the relation of a mile of, 6000 (geometric) feet, or a minute of arc on the earth surface, had been advanced as a universal measure for land and sea. The metric kilometre was selected to represent a centesimal minute of arc, on the same basis, with the circle divided into 400 degrees of 100 minutes.

3.7 *History of the Metric System*

The history of metric system is strictly connected with polar distance calculation. The metre (meter in American English), symbol m, is the fundamental unit of length in the International System of Units (SI). Originally intended to be one ten-millionth of the distance from the Earth's equator to the North Pole (at sea level), its definition has been periodically refined to reflect growing knowledge of metrology. Since 1983, it has been defined as "the length of the path travelled by light in vacuum during a time interval of 1/299,792,458 of a second".

The original "Sacred Cubit" was a unit of measure equal to 25 British inches, and also equal to one 10-millionth part of the distance between the North Pole and the center of the Earth. In 1790 Charles Talleyrand was sent to the Paris Academy of Sciences in order to help establish a new worldwide system of weights and measures meant to replace the English system of weights and measures that was in use all over the world at the time. This new French measuring system would be based upon a new unit of measure known as the "meter." The meter (from the Greek word "metron") was designed to be a counterfeit cubit, equal to one 10-millionth part of the distance between the North Pole and the Equator:

Cubit = 1/10,000,000th part of distance from N. Pole to Earth's Center;

Meter = 1/10,000,000th part of distance from N. Pole to Earth's Equator.

The original Sacred Cubit was a length equal to 25 English inches, or 7 "hands." The "hand" measure is still used today by people who raise horses, it is a length of just under 4 inches (3.58 inches to be exact), and is equal to the width of a man's hand, not including the thumb.

A decimal-based unit of length, the *universal measure* or *standard* was proposed in an essay of 1668 by the English cleric and philosopher John Wilkins. In 1675, the Italian scientist Tito Livio Burattini, in his work *Misura Universale,* used the phrase *metro cattolico* (lit. "catholic [i.e. universal] measure"), derived from the Greek *métron katholikón,* to denote the standard unit of length derived from a pendulum. In the wake of the French Revolution, a commission organised by the French Academy of Sciences and charged with determining a single scale for all measures, advised the adoption of a decimal system (27 October, 1790) and suggested a basic unit of length equal to one ten-millionth of the distance between the North Pole and the Equator, to be called 'measure' (*mètre*) (19th March 1791). The National Convention adopted the proposal in 1793. The first occurrence of *metre* in this sense in English dates to 1797.

4 THE FORMULA FOR THE PERIMETER OF AN ELLIPSE

The problem of calculating the distance from the equator to the pole basically comes down to calculate the perimeter of an ellipse and its quadrant. But rather strangely, the perimeter of an ellipse is very difficult to calculate!

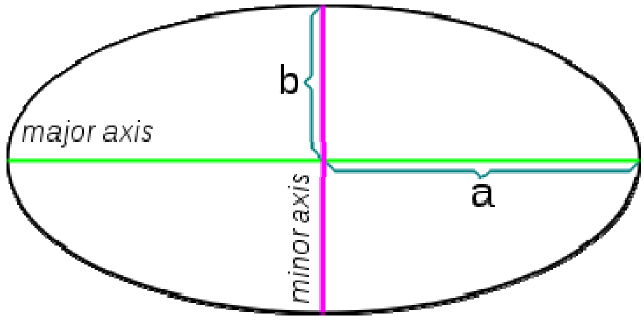

Figure 2. Ellipse parameter: a - major axis; b – minor axis

For an ellipse of Cartesian equation $x^2/a^2 + y^2/b^2 = 1$ with $a > b$:

- a is called the major radius or semimajor axis,
- b is the minor radius or semiminor axis,
- the quantity $e = (1 - b^2/a^2)^{1/2}$ is the eccentricity of the ellipse,
- the unnamed quantity $h = (a-b)^2 / (a+b)^2$ often pops up.

There is no simple exact formula to calculate perimeter of an ellipse. There are simple formulas but they are not exact, and there are exact formulas but they are not simple. Here, we'll discuss many approximations, and two exact expressions (infinite sums). There are many formulas, here are a few interesting ones only, but not all [Michon, 2012]:

Approximation 1

This approximation will be within about 5% of the true value, so long as *a* is not more than 3 times longer than *b* (in other words, the ellipse is not too "squashed"):

$$p \approx 2\pi\sqrt{\frac{a^2 + b^2}{2}} \tag{1}$$

Approximation 2

It is found in dictionaries and other practical references as a simple approximation to the perimeter *p* of the ellipse:

$$p \approx \pi\sqrt{2\left(a^2 + b^2\right) - \frac{(a-b)^2}{2}} \tag{2}$$

Approximation 3

An approximate expression, for *e* not too close to 1, is:

$$p \approx \pi\left[\frac{3}{2}\left(a+b\right) - \sqrt{ab}\right] \tag{3}$$

Approximation 4

The famous Indian mathematician S. Ramanujan in 1914 came up with this better approximation:

$$p \approx \pi\left[3(a+b)-\sqrt{(3a+b)(a+3b)}\right] \quad (4)$$

Approximation 5

The above Ramanujan formula is only about twice as precise as a formula proposed by Lindner between 1904 and 1920, which is obtained simply by retaining only the first three terms in an exact expansionin terms of *h* (these three terms happen to form a perfect square).

Firstly we must calculate "*h*":

$$h=\frac{(a-b)^2}{(a+b)^2} \quad (5)$$

$$p \approx \pi(a+b)[1+h/8]^2 \quad (6)$$

Approximation 6

A better 1914 formula, also due to Ramanujan, called Ramanujan II, gives the perimeter *p:*

$$p \approx \pi\ (a+b)\left[1+\frac{3h}{10+\sqrt{4-3h}}\right] \quad (7)$$

Approximation 7

R.G. Hudson is traditionally credited for a formula without square roots which he did not invent and which is intermediate in precision between the two Ramanujan formulas.

$$p \approx \pi\ (a+b)\frac{64-3h^2}{64-16h} \quad (8)$$

Approximation 8

A more precise Padé approximant consists of the optimized ratio of two quadratic polynomials of *h* and leads to the following formula:

$$p \approx \pi\ (a+b)\frac{256-48h-21h^2}{256-112h+3h^2} \quad (9)$$

Approximation 9

One more popular approximation, Peano's formula:

$$p \approx \pi(a+b)\frac{3-\sqrt{(1-h)}}{2} \quad (10)$$

Infinite Series 1

An exact expression of the perimeter *p* of an ellipse was first published in 1742 by the Scottish mathematician Colin Maclaurin.

This is an exact formula, but it requires an "infinite series" of calculations to be exact, so in practice we still only get an approximation.

Firstly we must calculate *e* (the "eccentricity", not Euler's number "e"):

$$e=\frac{\sqrt{a^2-\ b^2}}{a} \quad (11)$$

Then use this "infinite sum" formula:

$$p=2a\pi\left(1-\sum_{i=1}^{\infty}\frac{(2i)!^2}{(2^i\cdot i!)^4}\cdot\frac{e^{2i}}{2i-1}\right) \quad (12)$$

which may look complicated, but expands like this:

$$p=2a\pi\left[1-\left(\frac{1}{2}\right)^2 e^2-\left(\frac{1\cdot3}{2\cdot4}\right)^2\frac{e^4}{3}-\left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2\frac{e^6}{5}-\ldots\right] \quad (13)$$

The terms continue on infinitely, and unfortunately we must calculate a lot of terms to get a reasonably close answer.

Infinite Series 2

Author's favourite exact "infinite sum" formula (because it gives a very close answer after only a few terms) is as follows:

$$p=\pi(a+b)\sum_{n=0}^{\infty}\binom{0.5}{n}^2 h^n \quad (14)$$

Note: the $\binom{0.5}{n}$ is the binomial coefficient with half-integer factorials.

It may look a bit scary, but it expands to this series of calculations, now called the Gauss-Kummer series of *h*:

$$p=\pi(a+b)\left(1+\frac{1}{4}h+\frac{1}{64}h^2+\frac{1}{256}h^3+\ldots\right) \quad (15)$$

The more terms we calculate, the more accurate it becomes (the next term is $25h^4/16384$, which is getting quite small, and the next is $49h^5/65536$, then $441h^6/1048576$).

Comparison of the results of calculations done according to all the methods described above is shown in Table 1.

Table 1. Comparison of results of formulas for perimeter of an ellipse and its quadrant, for parameters a and b of ellipsoid WGS-84, where a = 6,378,137 m, b = 6,356,752.3142452 m

Method	Formula	Perimeter	Quadrant
Approximation 1	(1)	40,007,891.12054030	10,001,972.78013510
Approximation 2	(2)	40,007,862.91723600	10,001,965.72930900
Approximation 3	(3)	40,007,862.91726590	10,001,965.72931650
Approximation 4	Ramanujan I (4)	40,007,862.91725090	10,001,965.72931270
Approximation 5	Lindner (6)	40,007,862.91725100	10,001,965.72931270
Approximation 6	Ramanujan II (7)	40,007,862.91725100	10,001,965.72931270
Approximation 7	Hudson (8)	40,007,862.91726090	10,001,965.72931520
Approximation 8	Pade (9)	40,007,862.91725090	10,001,965.72931270
Approximation 9	Peano (10)	40,007,862.91726590	10,001,965.72931650
Infinite Series 1	Maclaurin (13)	40,007,862.91811430	10,001,965.72952860
Infinite Series 2	Gauss-Kummer (15)	40,007,862.91725100	10,001,965.72931270

5 MERIDIAN ARC

On any surface which fulfils the required continuity conditions, the shortest path between two points on the surface is along the arc of a geodesic curve. On the surface of a sphere the geodesic curves are the great circles and the shortest path between any two points on this surface is along the arc of a great circle, but on the surface of an ellipsoid of revolution, the geodesic curves are not so easily defined except that the equator of this ellipsoid is a circle and its meridians are ellipses [Williams, 1996].

In geodesy, a meridian arc measurement is a highly accurate determination of the distance between two points with the same longitude. Two or more such determinations at different locations then specify the shape of the reference ellipsoid which best approximates the shape of the geoid. This process is called the determination of the figure of the Earth. The earliest determinations of the size of a spherical Earth required a single arc. The latest determinations use astrogeodetic measurements and the methods of satellite geodesy to determine the reference ellipsoids.

5.1 *The Earth as an Ellipsoid*

High precision land surveys can be used determine the distance between two places at "almost" the same longitude by measuring a base line and a chain of triangles (suitable stations for the end points are rarely at the same longitude). The distance Δ along the meridian from one end point to a point at the same latitude as the second end point is then calculated by trigonometry. The surface distance Δ is reduced to Δ', the corresponding distance at mean sea level. The intermediate distances to points on the meridian at the same latitudes as other stations of the survey may also be calculated.

The geographic latitudes of both end points, φ_s (standpoint) and φ_f (forepoint) and possibly at other points are determined by astrogeodesy, observing the zenith distances of sufficient numbers of stars. If latitudes are measured at end points only, the radius of curvature at the mid-point of the meridian arc can be calculated from $R = \Delta'/(|\varphi_s - \varphi_f|)$. A second meridian arc will allow the derivation of two parameters required to specify a reference ellipsoid. Longer arcs with intermediate latitude determinations can completely determine the ellipsoid. In practice multiple arc measurements are used to determine the ellipsoid parameters by the method of least squares. The parameters determined are usually the semi-major axis, a, and either the semi-minor axis, b, or the inverse flattening $1/f$, (where the flattening is $f = (a-b)/a$).

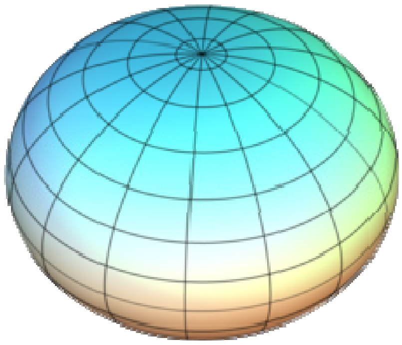

Figure 3. An oblate spheroid (ellipsoid)

5.2 *Meridian Distance on the Ellipsoid*

The determination of the meridian distance that is the distance from the equator to a point at latitude φ on the ellipsoid is an important problem in the theory of map projections, particularly the Transverse Mercator projection. Ellipsoids are normally specified in terms of the parameters defined above, a, b, $1/f$, but in theoretical work it is useful to define extra parameters, particularly the eccentricity, e, and the third flattening n. Only two of these parameters are independent and there are many relations between them [Banachowicz, 2006]:

$$f = \frac{a-b}{a}, \qquad e^2 = f(2-f), \qquad n = \frac{a-b}{a+b} = \frac{f}{2-f} \qquad (16)$$
$$b = a(1-f) = a\left(1-e^2\right)^{1/2}, \qquad e^2 = \frac{4n}{(1+n)^2}.$$

The radius of curvature is defined as

$$M(\varphi) = \frac{a\left(1-e^2\right)}{\left(1-e^2\sin^2\varphi\right)^{3/2}}, \qquad (17)$$

so that the arc length of an infinitesimal element of the meridian is $M(\varphi)d\varphi$ (with φ in radians). Therefore the meridian distance from the equator to latitude φ is

$$m(\varphi) = \int_0^{\varphi} M(\varphi)\, d\varphi = a\left(1-e^2\right)\int_0^{\varphi}\left(1-e^2\sin^2\varphi\right)^{-3/2} d\varphi. \quad (18)$$

The distance from the equator to the pole, the polar distance, is

$$m_p = m(\pi/2). \quad (19)$$

The above integral is related to a special case of an incomplete elliptic integral of the third kind.

$$m(\varphi) = a\left(1-e^2\right)\Pi(\varphi, \mathrm{e}, \mathrm{e}). \quad (20)$$

Many methods have been used for the computation of the integral of formula (18). All these methods and formula can be used for the calculation of the distance along the great elliptic arc by formula (21).

$$M_0^{\varphi} = \int_0^{\varphi} \frac{a(1-e^2)}{(1-e^2\sin\varphi)^{\frac{3}{2}}} d\varphi \quad (21)$$

Equation (21) can be transformed to an elliptic integral of the second type, which cannot be evaluated in a closed form. The calculation can be performed either by numerical integration methods, such as Simpson's rule, or by the binomial expansion of the denominator to rapidly converging series, retention of a few terms of these series and further integration by parts. This process yields results like formula (22).

$$M_0^{\varphi} = a\left(1-e^2\right)\left(\left(1+\frac{3}{4}e^2+\ldots\right)\varphi - \left(\frac{3}{8}e^2+\frac{15}{32}e^4\ldots\right)\sin 2\varphi + \ldots\right) \quad (22)$$

Equation (22) is the standard geodetic formula for the accurate calculation of the meridian arc length, which is proposed in a number of textbooks such as in Torge's *Geodesy* using up to $\sin(2\varphi)$ terms.

According to Snyder [Snyder, 1987] and Torge [Torge, 2001], Simpson's numerical integration of formula (21) does not provide satisfactory results and consequently the standard computation methods for the length of the meridian arc are based on the use of series expansion formulas, such as formula (22) and more detailed formulas presented below.

Delambre

The above integral may be approximated by a truncated series in the square of the eccentricity (approximately 1/150) by expanding the integrand in a binomial series. Setting $s = \sin\varphi$,

$$(1-e^2\sin^2\varphi)^{-3/2} = 1 + b_2e^2s^2 + b_4e^4s^4 + b_6e^6s^6 + b_8e^8s^8 + \ldots, \quad (23)$$

where

$$b_2 = \frac{3}{2}, \qquad b_4 = \frac{15}{8} \qquad b_6 = \frac{35}{16}, \qquad b_8 = \frac{315}{128}.$$

Using simple trigonometric identities the powers of $\sin\varphi$ may be reduced to combinations of factors of $\cos 2p\varphi$. Collecting terms with the same cosine factors and integrating gives the following series, first given by Delambre in 1799.

$$m(\varphi) = A_0\varphi + A_2\sin 2\varphi + A_4\sin 4\varphi + A_6\sin 6\varphi + A_8\sin 8\varphi + \ldots, \quad (24)$$

where:

$$A_0 = a\left(1-e^2\right)\left(1+\frac{3}{4}e^2+\frac{45}{64}e^4+\frac{175}{256}e^6+\frac{11025}{16384}e^8\right)$$

$$A_2 = -\frac{a\left(1-e^2\right)}{2}\left(\frac{3}{4}e^2+\frac{15}{16}e^4+\frac{525}{512}e^6+\frac{2205}{2048}e^8\right)$$

$$A_4 = \frac{a\left(1-e^2\right)}{4}\left(\frac{15}{64}e^4+\frac{105}{256}e^6+\frac{2205}{4096}e^8\right)$$

$$A_6 = -\frac{a\left(1-e^2\right)}{6}\left(\frac{35}{512}e^6+\frac{315}{2048}e^8\right)$$

$$A_8 = \frac{a\left(1-e^2\right)}{8}\left(\frac{315}{16384}e^8\right)$$

The numerical values for the semi-major axis and eccentricity of the WGS-84 ellipsoid give, in metres,

$$m(\varphi) = 6367449.146\varphi - 16038.509\sin 2\varphi + 16.833\sin 4\varphi - 0.022\sin 6\varphi + 0.00003\sin 8\varphi \quad (25)$$

The first four terms have been rounded to the nearest millimetre whilst the eighth order term gives rise to sub-millimetre corrections. Tenth order series are employed in modern "wide zone" implementations of the Transverse Mercator projection.

For the WGS-84 ellipsoid the distance from equator to pole is given (in metres) by

$$m_p = \frac{1}{2}\pi A_0 = 10\,001\,965.729\,m.$$

The third flattening n is related to the eccentricity by

$$e^2 = \frac{4n}{(1+n)^2} = 4n\left(1-2n+3n^2-4n^3+\ldots\right). \quad (26)$$

With this substitution the integral for the meridian distance becomes

$$m(\varphi) = \int_0^{\varphi} \frac{a(1-n)^2(1+n)}{\left(1+2n\cos 2\varphi+n^2\right)^{3/2}} d\varphi. \quad (27)$$

This integral has been expanded in several ways, all of which can be related to the Delambre series.

Bessel's formula

In 1837 Bessel expanded this integral in a series of the form:

$$m(\varphi)=a(1-n)^2(1+n)[D_0\varphi-D_2\sin 2\varphi+D_4\sin 4\varphi-D_6\sin 6\varphi+\ldots], \tag{28}$$

where

$$D_0=1+\frac{9}{4}n^2+\frac{225}{64}n^4+\ldots, \qquad D_4=\frac{15}{16}n^2+\frac{105}{64}n^4+\ldots,$$

$$D_2=\frac{3}{2}n+\frac{45}{16}n^3+\frac{525}{128}n^5+\ldots, \quad D_6=\frac{35}{48}n^3+\frac{315}{256}n^5+\ldots,$$

Since n is approximately one quarter of the value of the squared eccentricity, the above series for the coefficients converge 16 times as fast as the Delambre series.

Helmert's formula

In 1880 Helmert extended and simplified the above series by rewriting

$$(1-n)^2(1+n)=\frac{1}{1+n}\left(1-n^2\right)^2 \tag{29}$$

and expanding the numerator terms.

$$m(\varphi)=\frac{a}{1+n}[H_0\varphi-H_2\sin 2\varphi+H_4\sin 4\varphi-H_6\sin 6\varphi+H_8\sin 8\varphi+\ldots] \tag{30}$$

with

$$H_0=1+\frac{n^2}{4}+\frac{n^4}{64}+\ldots \qquad H_6=\frac{35}{48}n^3+\ldots$$

$$H_2=\frac{3}{2}\left(n-\frac{n^3}{8}+\ldots\right) \qquad H_8=\frac{315}{512}n^4+\ldots$$

$$H_4=\frac{15}{16}\left(n^2-\frac{n^4}{4}+\ldots\right)$$

UTM

Despite the simplicity and fast convergence of Helmert's expansion the U.S. DMA adopted the fully expanded form of the Bessel series reported by Hinks in 1927. This expansion is important, despite the poorer convergence of series in n, because it is used in the definition of UTM [Bowring, 1983].

$$m(\varphi)=B_0\varphi+B_2\sin 2\varphi+B_4\sin 4\varphi+B_6\sin 6\varphi+B_8\sin 8\varphi+\ldots, \tag{31}$$

where the coefficients are given to order n^5 by

$$B_0=a\left(1-n+\frac{5}{4}n^2-\frac{5}{4}n^3+\frac{81}{64}n^4-\frac{81}{64}n^5+\ldots\right),$$

$$B_2=-\frac{3}{2}a\left(n-n^2+\frac{7}{8}n^3-\frac{7}{8}n^4+\frac{55}{64}n^5-\ldots\right),$$

$$B_4=\frac{15}{16}a\left(n^2-n^3+\frac{3}{4}n^4-\frac{3}{4}n^5+\ldots\right),$$

$$B_6=-\frac{35}{48}a\left(n^3-n^4+\frac{11}{16}n^5-\ldots\right),$$

$$B_8=\frac{315}{512}a\left(n^4-n^5+\ldots\right),$$

Generalized series:

The above series, to eighth order in eccentricity or fourth order in third flattening, are adequate for most practical applications. Each can be written quite generally. For example, Kazushige Kawase (2009) derived following general formula [Kawase, 2011]:

$$m(\varphi)=\frac{a}{1+n}\cdot \sum_{j=0}^{\infty}\left(\prod_{k=1}^{j}\varepsilon_k\right)^2\left\{\varphi+\sum_{\ell=1}^{2j}\left(\frac{1}{\ell}-4\ell\right)\sin 2\ell\varphi\prod_{m=1}^{\ell}\varepsilon_{j+(-1)^m m/2}^{(-1)^m}\right\} \tag{32}$$

where

$$\varepsilon_i=\frac{3n}{2i}-n.$$

Truncating the summation at j = 2 gives Helmert's approximation.

The polar distance may be approximated by the Thomas Muir's formula:

$$m_p=\int_0^{\pi/2}M(\varphi)\,d\varphi\approx\frac{\pi}{2}\left[\frac{a^{3/2}+b^{3/2}}{2}\right]^{2/3}. \tag{33}$$

6 EXISTING METHODS FOR THE SOLUTION OF THE GREAT ELLIPTIC SAILING

6.1 *Bowring Method for the Direct and Inverse Solutions for the Great Elliptic Line*

Bowring [Bowring, 1984] provides formulas for the solution of the direct and inverse great elliptic sailing problem. Bowring's formulas can be used for the calculations of the great elliptic arc length and the forward and backward azimuths.

The method of Bowring for the calculation of great elliptic arc length employs the use of an auxiliary geodetic sphere and various types of coordinates, such as, geodetic, geocentric, Cartesian and polar. These formulas for the great elliptic distance have been tested and it was proved that they provide very satisfactory results in terms of obtained accuracy. Nevertheless other simpler computations methods of the length of the great elliptic arc can be used by the employment of standard geodetic formulas for the length of the arc of the meridian, after the proper modification of the parameters of the meridian ellipse with those of the great ellipse, such as formula (21). The formulas used by Bowring for the calculation of the forward and backward azimuths, unlike those for the distance, are very much simpler than other methods of the same accuracy [Pallikaris & Latsas, 2009].

6.2 *William's Method for the Computation of the Distance Along the Great Elliptic Arc*

Williams [Williams, 1996] provides formulas for the computation of the sailing distance along the arc of the great ellipse. These formulas have the general form of the integral of formula (21). For the computation of the eccentricity e_{ge} and the geodetic great elliptic angle φ_{ge} of formula (21), Williams provides simple and compact formulas. For the evaluation of this integral Williams employs the cubic spline integration method of Phythian and Williams [Phythian & Williams, 1985].

6.3 *Earle's Method for Vector Solutions*

Earle [Earle, 2000] has proposed a method of computing distance along a great ellipse that allows the integral for distance to be computed directly using the built-in capabilities of commercial mathematical software. This obviates the need to write code in arcane computer languages. According to Earle, his method has been prepared with the syntax of a particular commercial mathematics package in mind.

6.4 *Walwyn's Great Ellipse Algorithm*

Walwyn [Walwyn, 1999] presented an algorithm for the computation of the arc length along the great ellipse and the initial heading to steer. The algorithm uses various formulas for the calculation of distance and azimuths (courses). In some cases, probably for the sake of simplicity, these formulas are not the right ones used in standard geodetic computations, as the formulas for the transformation of the geodetic latitudes to geocentric.

6.5 *The Pallikaris and Latsas's New Algorithm for the Great Elliptic Sailing*

Algorithm proposed by Pallikaris and Latsas [Pallikaris & Latsas, 2009] was initially developed as a supporting tool in another research work of the Pallikaris on the implementation of sailing calculations in GIS-based navigational systems (ECDIS and ECS). The complete great elliptic sailing problem is solved including, in addition to the great elliptic arc distance, the geodetic coordinates of an unlimited number of intermediate points along the great elliptic arc. The algorithm has been developed having a mind to avoid the use of advanced numerical methods, in order to allow for the convenient implementation even in programmable pocket calculators.

The algorithm starts with the calculation of the eccentricity of the great ellipse and the geocentric and geodetic great elliptic angles of the points of departure and destination. For this part of the algorithm we used the formulas proposed by Williams [Williams, 1996] because they are simple, straightforward and provide accurate results. For the calculation of the length of the great elliptic arc we used the standard geodetic series expansion formulas for the meridian arc length that are presented in basic geodesy textbooks like [Torge, 2001] after their proper modification for the great ellipse.

Calculations of the Great Elliptic Distance:

Length of the great elliptic arc:

$$S_{12} = \int_{\varphi_{ge1}}^{\varphi_{ge2}} \frac{a\left(1-e_{ge}^2\right)}{\sqrt{\left(1-e_{ge}^2\sin^2\left(\varphi_{ge}\right)\right)^3}}\, d\chi \approx$$
$$a\left(1-e^2\right)\left(\left(1+\frac{3}{4}e^2+\dots\right)\varphi_{ge}-\left(\frac{3}{8}e^2+\frac{15}{32}e^4\dots\right)\sin 2\varphi_{ge}+\dots\right)_{\varphi_{ge1}}^{\varphi_{ge2}} \quad (34)$$

up to $\sin(8\varphi)$ terms.

6.6 *The Snyder's Series Approximations for the Meridian Ellipse*

Equation 21 is easily evaluated numerically and even elementary methods such as Simpson's rule will work but may not have sufficient precision, although an algorithm described in [Williams, 1998] is known to work well. It is preferable however, to use an adaptive algorithm that adjusts the intervals of the integrand according to the slope of the function.

The function $f_1(\varphi)$ below is a compact harmonic series approximation to equation 21 for meridional distance [Snyder, 1987].

$$f_1(\varphi) = a\left[a_0\varphi + \sum_{n=1}^{3} a_n \sin(2n\varphi)\right] \quad (35)$$

The coefficients are:

$$a_0 = 1 - \frac{1}{4}\varepsilon^2 - \frac{3}{64}\varepsilon^4 - \frac{5}{256}\varepsilon^6$$
$$a_1 = -\left(\frac{3}{8}\varepsilon^2 + \frac{3}{32}\varepsilon^4 + \frac{45}{1024}\varepsilon^6\right)$$
$$a_2 = \left(\frac{15}{256}\varepsilon^4 + \frac{45}{1024}\varepsilon^6\right)$$
$$a_3 = -\left(\frac{35}{3072}\varepsilon^6\right)$$

Distance M_{12} between two latitudes on the meridional arc in the same hemisphere can be determined using equation 20 i.e.

$$M_{12} = f_1(\varphi_2) - f_1(\varphi_1) \quad (36)$$

Loss of significant digits is reduced for small angular separations if differencing is applied to equation 20 resulting in:

$$M_{12} = a\left[a_0(\varphi_2-\varphi_1) + \sum_{n=1}^{3} 2a_n \cos\left(n(\varphi_2+\varphi_1)\right)\sin\left(n(\varphi_2-\varphi_1)\right)\right] \quad (37)$$

which will be adapted later to give distance on the great ellipse. There is also a companion harmonic inversion series to equation 35, described by Snyder and attributed to an earlier work [Adams, 1921] that used the Lagrange Inversion Theorem to construct the inversion series. It provides geodetic latitude as a function of normalized meridional distance. The condensed form of this harmonic inversion series is:

$$f_2(u) = b_0 u + \sum_{n=1}^{4} b_n \sin(2nu) \qquad (38)$$

the constants for which are:

$$b_0 = 1$$
$$b_1 = \frac{3}{2}\varepsilon_1 - \frac{27}{32}\varepsilon_1^3$$
$$b_2 = \frac{21}{16}\varepsilon_1^2 - \frac{55}{332}\varepsilon_1^4$$
$$b_3 = \frac{151}{96}\varepsilon_1^3$$
$$b_4 = \frac{1097}{512}\varepsilon_1^4$$

and

$$\varepsilon_1 = \left(1-\sqrt{\alpha}\right)/\left(1+\sqrt{\alpha}\right)$$

For each value of the normalized distance $u = \frac{\pi}{2}\frac{M}{M_0}$, the function $f_2(u)$ returns a value of geodetic latitude φ corresponding to the given meridional distance *M*. The constant M_0 is the meridional distance from the equator to the pole i.e. $M_0 = f_1\left(\frac{\pi}{2}\right)$ or, equivalently, $M_0 = a\left(a_0 \pi / 2\right)$. Both of these series are periodic and can be used over arcs spanning any interval in the range $0 < \varphi < 2\pi$ [Earle, 2011].

6.7 *The Deakin's Meridian Distance M*

Meridian distance *M* is defined as the arc of the meridian ellipse from the equator to the point of latitude φ.

This is an elliptic integral that cannot be expressed in terms of elementary functions; instead, the integrand is expanded by into a series using Taylor's theorem then evaluated by term-by-term integration. The usual form of the series formula for *M* is a function of φ and powers of e^2 obtained from [Deakin & Hunter, 2010], [Deakin, 2012]

$$M = a\left(1-e^2\right)\int_0^{\varphi}\left(1-e^2\sin^2\varphi\right)^{-3/2} d\varphi \qquad (39)$$

But the German geodesist F.R. Helmert (1880) gave a formula for meridian distance as a function of φ and powers of *n* that required fewer terms for the same accuracy. Helmert's method of development is given in [Deakin & Hunter, 2010] and with some algebra we may write

$$M = \frac{a}{1+n}\int_0^{\varphi}\left(1-n^2\right)^2\left(1+n^2+2n\cos 2\varphi\right)^{-3/2} d\varphi \qquad (40)$$

It can be shown, using Maxima, that (39) and (40) can easily be evaluated and *M* written as

$$M = a\left(1-e^2\right)\begin{pmatrix} b_0\varphi + b_2 sin2\varphi + b_4 sin4\varphi + \\ b_6 sin6\varphi + b_8 sin8\varphi + b_{10} sin10\varphi + \ldots \end{pmatrix} \qquad (41)$$

where the coefficients $\{b_n\}$ are to order e^{10} as follows:

$$b_0 = 1 + \frac{3}{4}e^2 + \frac{45}{64}e^4 + \frac{175}{256}e^6 + \frac{11025}{16384}e^8 + \frac{43659}{65536}e^{10} + \ldots$$
$$b_2 = -\frac{3}{8}e^2 - \frac{15}{32}e^4 - \frac{525}{1024}e^6 - \frac{2205}{4096}e^8 - \frac{72765}{131072}e^{10} - \ldots$$
$$b_4 = \frac{15}{256}e^4 + \frac{105}{1024}e^6 + \frac{2205}{16384}e^8 + \frac{10395}{65536}e^{10} + \ldots$$
$$b_6 = -\frac{35}{3072}e^6 - \frac{105}{4096}e^8 - \frac{10395}{262144}e^{10} - \ldots$$
$$b_8 = \frac{315}{131072}e^8 + \frac{3465}{524288}e^{10} + \ldots$$
$$b_{10} = -\frac{693}{1310720}e^{10} - \ldots$$

or

$$M = \frac{a}{1+n}\begin{Bmatrix} c_0\varphi + c_2 sin2\varphi + c_4 sin4\varphi + \\ c_6 sin6\varphi + c_8 sin8\varphi + c_{10} sin10\varphi + \ldots \end{Bmatrix} \qquad (42)$$

where the coefficients $\{c_n\}$ are to order n^5 as follows

$$c_0 = 1 + \frac{1}{4}n^2 + \frac{1}{64}n^4 + \ldots$$
$$c_2 = -\frac{3}{2}n + \frac{3}{16}n^3 + \frac{3}{128}n^5 + \ldots$$
$$c_4 = \frac{15}{16}n^2 - \frac{15}{64}n^4 + \ldots$$
$$c_6 = -\frac{35}{48}n^3 + \frac{175}{768}n^5 + \ldots$$
$$c_8 = \frac{315}{512}n^4 + \ldots$$
$$c_{10} = -\frac{693}{1280}n^5 + \ldots$$

Note here that for WGS-84 ellipsoid, where *a* = 6,378,137 m and *f* = 1/298.257223563 the ellipsoid constants $n = \mathrm{n} = 1.679220386383705\mathrm{e}-003$ and $e^2 = 6.694379990141317\mathrm{e}-003$, and $n^5 \cong \frac{e^{10}}{1007} \cong \frac{e^{12}}{6.7}$ [Williams, 2002], [Deakin, 2012].

This demonstrates that the series (42) with fewer terms in the coefficients $\{c_n\}$ is at least as 'accurate' as the series (41). To test this consider the meridian distance expressed as a sum of terms $M = M_0 + M_2 + M_4 + \ldots$, where for series (41)

$$M_0 = a\left(1-e^2\right)b_0\varphi,\ M_2 = a\left(1-e^2\right)b_2 sin2\varphi, M_4 = a\left(1-e^2\right)b_4 sin4\varphi, \text{ etc.}$$

and for series (42)

$$M_0 = \frac{a}{1+n}c_0\varphi,\ M_2 = \frac{a}{1+n}c_2\sin 2\varphi,\ M_4 = \frac{a}{1+n}c_4\sin 4\varphi, \text{ etc.}$$

Maximum values for $M_0, M_2, M_4 \ldots$ occur at latitudes $\varphi = 90°, 45°, 22.5°, \ldots$ when $\varphi =$ max or $\sin k\varphi = 1$ and testing the differences between terms at these maximums revealed no differences greater than 0.5 micrometres. So series (42) should be the preferable method of computation. Indeed, further truncation of the coefficients $\{c_n\}$ to order n^4 and truncating series (42) at $c_8\sin 8\varphi$ revealed no differences greater than 1 micrometre [Deakin, 2012].

Quadrant Length Q

The quadrant length of the ellipsoid *Q* is the length of the meridian arc from the equator to the pole and is obtained from equation (41) by setting $\varphi = \frac{1}{2}\pi$, and noting that $sin2\varphi, sin4\varphi, sin6\varphi$ all equal zero, giving

$$Q = a\left(1-e^2\right)b_{0\frac{\pi}{2}} = a\left(1-e^2\right)\left\{1+\frac{3}{4}e^2+\frac{45}{64}e^4+\frac{175}{256}e^6+\frac{11025}{16384}e^8+\frac{43659}{65536}e^{10}+\ldots\right\}\frac{\pi}{2} \tag{43}$$

Similarly, using equation (42)

$$Q = \frac{a}{1+n}c_{0\frac{\pi}{2}} = \frac{a}{1+n}\left\{1+\frac{1}{4}n^2+\frac{1}{64}n^4+\ldots\right\}\frac{\pi}{2} \tag{44}$$

7 GEODETIC FORMULAS FOR THE MERIDIAN ARC LENGTH

7.1 *The Snyder's Series Approximations for the Meridian Ellipse*

The methods and formulas used to calculate the length of the arc of the meridian for precise sailing calculations on the ellipsoid, such as "rhumb-line sailing", "great elliptic sailing" and "geodesic sailing" are simplified forms of general geodetic formulas used in geodetic applications [Pallikaris, Tsoulos, Paradissis, 2009]. In this section an overview of the most important geodetic formulas along with general comments and remarks on their use is carried out. For consistency purposes and in order to avoid confusion in certain formulas the symbolization has been changed from that of the original sources.

Equation (21) can be transformed to an elliptic integral of the second type, which cannot be evaluated in a "closed" form. The calculation can be performed either by numerical integration methods, such as Simpson's rule, or by the binomial expansion of the denominator to rapidly converging series, retention of a few terms of these series and further integration by parts. According to Snyder [Snyder, 1987] and Torge [Torge, 2001], Simpson's numerical integration does not provide satisfactory results and consequently the standard computation methods are based on the use of series expansion formulas. Expanding the denominator of (21) by the binomial theorem yields:

$$M_0^{\varphi} = a\cdot\left(1-e^2\right)\int_0^{\varphi}\left(1+\frac{3}{2}e^2 sin^2\varphi+\frac{15}{8}e^4 sin^4\varphi+\frac{35}{16}e^6 sin^6\varphi\right)dx \tag{45}$$

Since the values of powers of *e* are very small, equation (45) is a rapidly converging series. Integrating (45) by parts we obtain:

$$M_0^{\varphi} = a\left(1-e^2\right)\left(\begin{array}{l}\left(1+\frac{3}{4}e^2+\ldots\right)\varphi-\left(\frac{3}{8}e^2+\frac{15}{32}e^4\ldots\right)\sin 2\varphi+\\ \left(\frac{15}{256}e^4+\frac{105}{1024}e^6+\ldots\right)\sin 4\varphi+\ldots\end{array}\right) \tag{46}$$

Equation (46) is the standard geodetic formula for the accurate calculation of the meridian arc length, which is proposed in a number of textbooks such as in Torge's "Geodesy" using up to sin(2φ) terms, [Torge, 2001] and in Veis' "Higher Geodesy" using up to sin(*8φ*) terms [Veis, 1992]. A rigorous derivation of (46) for terms up to sin(6φ), is presented in [Pearson, 1990].

Equation (46) can be written in the form of equation (47) provided by Veis [Veis, 1992]

$$M_0^{\varphi} = a\left(1-e^2\right)\left(M_0\varphi - M_2 2\varphi + M_4 4\varphi - M_6 6\varphi + M_8 8\varphi + \ldots\right) \tag{47}$$

$$M_0 = 1+\frac{3}{4}e^2+\frac{45}{64}e^4+\frac{175}{256}e^6+\frac{11025}{16384}e^8+\ldots$$
$$M_2 = \frac{3}{8}e^2+\frac{15}{32}e^4+\frac{525}{1024}e^6+\frac{2205}{4096}e^8+\ldots$$
$$M_4 = \frac{15}{256}e^4+\frac{105}{1024}e^6+\frac{2205}{16384}e^8+\ldots$$
$$M_6 = \frac{35}{3072}e^6+\frac{315}{12288}e^8+\ldots$$
$$M_8 = \frac{315}{131072}e^8+\ldots$$

Equation (48) is derived directly from equation (47) for the direct calculation of the length of the meridian arc between two points (A and B) with latitudes φ_A and φ_B. In the numerical tests for the assessment of the relevant errors of selected alternative formulas, we will refer to equations (47) and (48) as the "Veis - Torge" formulas.

$$M_{\varphi_A}^{\varphi_B} = a\left(1-e^2\right)[M_0\left(\varphi_A-\varphi_B\right)- M_2(\sin 2\varphi_B - \sin 2\varphi_A) + M_4(\sin 4\varphi_B - \sin 4\varphi_A) - M_6\left(\sin 6\varphi_B - \sin 6\varphi_A\right) + M_8(\sin 8\varphi_B - \sin 8\varphi_A)] \quad (48)$$

Equations (47) and (48) are the basic series expansion formulas used for the calculation of the meridian arc. They are rapidly converging since the value of the powers of e is very small. In most applications, very accurate results are obtained by formula (47) and the retention of terms up to $\sin(6\varphi)$ or $\sin(4\varphi)$ and 8th or 10th powers of e.

For sailing calculations on the ellipsoid it is adequate to retain only up to $\sin(2\varphi)$ terms, whereas for other geodetic applications it is adequate to retain up to $\sin(4\varphi)$ or $\sin(6\varphi)$ terms. The basic formulas (47) and (48) can be further manipulated and transformed to other forms. The most common of these forms is formula (49). Simplified versions of (49) (retaining up to A_6 and e^6 terms only) are proposed in textbooks such as in Bomford's "Geodesy" [Bomford, 1985].

$$M_0^{\varphi} = a\left(A_0\varphi - A_2 2\varphi + A_4 4\varphi - A_6 6\varphi + A_8 8\varphi \ldots\right) \quad (49)$$

$$A_0 = 1 - \frac{1}{4}e^2 - \frac{3}{64}e^4 - \frac{5}{256}e^6 - \frac{175}{16384}e^8 \ldots$$
$$A_2 = \frac{3}{8}\left(e^2 + \frac{1}{4}e^4 + \frac{1}{15}e^6 + \frac{35}{512}e^8 \ldots\right)$$
$$A_4 = \frac{15}{256}\left(e^4 + \frac{3}{4}e^6 + \frac{35}{64}e^8 \ldots\right)$$
$$A_6 = \frac{35}{3072}e^6 + \frac{175}{12228}e^8 \ldots$$
$$A_8 = \frac{315}{131072}e^8 \ldots$$

In the "Admiralty Manual of Navigation" [AMN, 1987] for the same formula (49) there are mentioned a little different coefficients (A_2 in particular):

$$A_0 = 1 - \frac{1}{4}e^2 - \frac{3}{64}e^4 - \frac{5}{256}e^6 - \ldots$$
$$A_2 = \frac{3}{8}\left(e^2 + \frac{1}{4}e^4 + \frac{15}{128}e^6 + \ldots\right)$$
$$A_4 = \frac{15}{256}\left(e^4 + \frac{3}{4}e^6 + \ldots\right)$$
$$A_6 = \frac{35}{3072}e^6 \ldots$$

Another formula for the meridian arc length is equation (50), which is used by Bowring [Bowring, 1983] as the reference for the derivation of other formulas, employing polar coordinates and complex numbers. The basic difference of formula (50) from (47), (48) and (49) is that (50) uses the ellipsoid parameters (a, b), instead of the parameters (a, e) which are used in formulas (47), (48) and (49).

$$M_0^{\varphi} = A_1\left(\varphi - B_1\frac{3}{2}n\sin 2\varphi - \frac{15}{16}n^2\sin 4\varphi + \frac{35}{48}n^3\sin 6\varphi - \frac{315}{512}n^4\sin 8\varphi + \ldots\right) \quad (50)$$

$$A_1 = \frac{a(1+\frac{1}{8}n^2)^2}{1+n}$$
$$B_1 = 1 - \frac{3}{8}n^2$$
$$n = \frac{a-b}{a+b} .$$

Bowring [Bowring, 1985] proposed also formula (51) for precise rhumb-line (loxodrome) sailing calculations. This formula calculates the meridian arc as a function of the mean latitude φ_m and the latitude difference $\Delta\varphi$ of the two points defining the arc on the meridian.

$$M_{\varphi_A}^{\varphi_B} = a(A_0\Delta\varphi - A_2\cos\left(2\varphi_m\right)\sin\left(\Delta\varphi\right) + A_4\cos\left(4\varphi_m\right)\sin\left(2\Delta\varphi\right) - A_6\cos\left(6\varphi_m\right)\sin\left(3\Delta\varphi\right) + A_8\cos\left(8\varphi_m\right)\sin\left(4\Delta\varphi\right) \quad (51)$$

In (51), the coefficients A_0, A_2, A_4, A_6, and A_8 are the same as in (49). Equation (49) has the general form of equation (52).

$$\Delta M = k_0\Delta\varphi - k_2\cos\left(2\varphi_m\right)\sin\left(\Delta\varphi\right) + k_4\cos\left(4\varphi_m\right)\sin\left(2\Delta\varphi\right) - k_6\cos\left(6\varphi_m\right)\sin\left(3\Delta\varphi\right) + k_8\cos\left(8\varphi_m\right)\sin\left(4\Delta\varphi\right) \quad (52)$$

In (52), the coefficients k_0, k_2, k_4, k_6, k_8 are: k_0= a A_0, k_2= a A_2, k_4= a A_4, k_6= a A_6, k_8= a A_8

7.2 *The Proposed New Formulas by Pallikaris, Tsoulos and Paradissis*

The proposed new formulas for the calculation of the length of the meridian in sailing calculations on the WGS-84 ellipsoid in meters and international nautical miles are (53) and (54), respectively [Pallikaris, et al, 2009].

$$M_{\varphi_A}^{\varphi_B} = 111132.95251\cdot\Delta\varphi - 16038.50861\cdot\left(\sin\left(\frac{\varphi_B\cdot\pi}{90}\right) - \sin\left(\frac{\varphi_A\cdot\pi}{90}\right)\right) \quad (53)$$

$$M_{\varphi_A}^{\varphi_B} = 60.006994\cdot\Delta\varphi - 8.660102\cdot\left(\sin\left(\frac{\varphi_B\cdot\pi}{90}\right) - \sin\left(\frac{\varphi_A\cdot\pi}{90}\right)\right) \quad (54)$$

- In both formulas (53) and (54) the values of geodetic latitudes φ_A and φ_B are in degrees and the calculated meridian arc length in meters and international nautical miles respectively. Formulas (53) and (54) have been derived from (48) for the WGS-84, since the geodetic datum employed in Electronic Chart Display and Information Systems is WGS-84. The derivation of the proposed formulas is based on the calculation of the M_0 and M_2 terms of (48) using up to the 8th power of e. This is equivalent to the accuracy provided by (49) using A_0 and A_2 terms with subsequent e terms extended up to the 10th power since in formula (48) the terms M_0, M_2, M_4 … are multiplied by $(1-e^2)$. According to the numerical tests carried out, which are presented in the next section, the proposed formulas have the following advantages:
- they are much simpler than and more than twice as fast as traditional geodetic methods of the same accuracy.
- they provide extremely high accuracies for the requirements of sailing calculations on the ellipsoid.

7.3 *The Author's Proposal*

Taking into account that the polar distance for WGS-84 is 10001965,7293127 m (see: Table 1) the author proposes some modification to the formula (53) proposed by Pallikaris, Tsoulos and Paradissis:

$$M_{\varphi_A}^{\varphi_B} = 6367449.1458234 \cdot \Delta\varphi - 16038.50862 \cdot \left(\sin\left(2\varphi_B\right) - \sin\left(2\varphi_A\right)\right) \qquad (55)$$

with φ in radians, and result in meters).

This formula will be a little bit more accurate than formula (53).

7.4 *Numerical Tests and Comparisons*

The different formulas and methods for the calculation of meridian arc distances, which have been initially evaluated and compared, are:
- the proposed new formulas by Pallikaris, Tsoulos and Paradissis (53) and (54), with author's modification (55) ;
- "Veis –Torge" formulas (formulas (47) and (48)) in various versions, according to the number of retained terms (1st version with up to M8 terms, 2nd version up to M6 terms, 3rd version up to M4 terms, 4th version up to M2 terms);
- The Bowring [Bowring, 1983] formula (50);
- The Bowring [Bowring, 1985] formula (51);

These numerical tests and comparisons have been based on the analysis of the calculations of the length of the polar distance. The results of the evaluated formulas are shown in Table 2.

It is not surprise that they correspond to the results presented in Table 1.

Table 2. Comparison of results of the calculations of polar distance for ellipsoid WGS-84 on the base of meridian distance formulas

Method	Formula	Quadrant
Deakin, 2010	(44)	10,001,965.72931270
Veis-Torge	(48)	10,001,965.72922300
Bomford, 1985	(49)	10,001,965.72931360
AMN, 1987	(49)	10,001,965.72952860
Bowring, 1983	(50)	10,001,965.72931270
Pallikaris, et al, 2009	(53)	10,001,965.72590000
Weintrit, 2013	(55)	10,001,965.72931270

The proposed new formulas by Pallikaris, Tsoulos and Paradissis [Pallikaris, et al, 2009] for the calculation of the meridian arc are sufficiently precise for sailing calculations on the ellipsoid. Higher sub metre accuracies can be obtained by the use of more complete equations with additional higher order terms. Seeking this higher accuracy for sailing calculations does not have any practical value for marine navigation and simply adds more complexity to the calculations only. In other than navigation applications, where higher sub metre accuracy is required, the Bowring formulas showed to be approximately two times faster than alternative geodetic formulas of similar accuracy.

8 CONCLUSIONS

Now we can surely state that for the WGS-84 ellipsoid of revolution the distance from equator to pole is 10,001,965.729 m, which was confirmed by a number of geometric and geodesic calculations presented in the paper.

The proposed formulas can be immediately used not only for the development of new algorithms for sailing calculations, but also for the simplification of existing algorithms without degrading the accuracies required for precise navigation. The simplicity of the proposed method allows for its easy implementation even on pocket calculators for the execution of accurate sailing calculations on the ellipsoid.

Original contribution affects and verifies established views based on approximated computational procedures used in the software of marine navigational systems and devices. Current stage of knowledge enables to implement geodesics based computations which present higher accuracy. It also lets to assess the quality of contemporary algorithms used in practical marine applications. It should be noted that an important step in the solution is simplification by the omission of the expansion part into power series of mathematical solutions, previously known from the literature, i.e., [Torge, 2001] and [Veis, 1992], and reliance in the explanatory memorandum of application, in particular, on the amount of the available processing power of modern calculating machine (processor). In the author's opinion this criterion is relevant from a practical point of view, but temporary, given the growth and availability of computing power, including GIS [Pallikaris et al., 2009], [Weintrit & Kopacz, 2011, 2012].

Scientific workshop employed to solve the problem makes use of various tools, i.e. of differential geometry, marine geodesy (marine navigation), analysis of measurement error, approximation theory and problems of modelling and computational complexity, mathematical and descriptive statistics, mathematical cartography. Geometrical problems are important aspect of the tested models which are used as the basis of calculations and solutions implemented in contemporary navigational devices and modern electronic chart systems.

This paper was written with a variety of readers in mind, ranging from practising navigators to theoretical analysts. It was also the author's goal to present current and uniform approaches to sailing calculations highlighting recent developments. Much insight may be gained by considering the examples. The algorithms applied for navigational purposes, in particular in ECDIS, should inform the user on actually used mathematical model and its limitations. The shortest distance (geodesics) between the points depends on the type of metric we use on the considered surface in general navigation. It is also important to know how the distance between two points on considered structure is determined.

An attempt to calculate the exact distance from the equator to the pole was just an excuse to look more closely at the methods of determining the meridian arc distance and the navigation calculations in general.

The navigation based on geodesic lines and connected software of the ship's devices (electronic chart, positioning and steering systems) gives a strong argument to research and use geodesic-based methods for calculations instead of the loxodromic trajectories in general. The theory is developing as well what may be found in the books on geometry and topology. This should motivate us to discuss the subject and research the components of the algorithm of calculations for navigational purposes.

Algorithms for the computation of geodesics on an ellipsoid of revolution are given. These provide accurate, robust, and fast solutions to the direct and inverse geodesic problems and they allow differential and integral properties of geodesics to be computed [Karney, 2011] and [Karney, 2013].

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