TransNav
http://www.transnav.eu
the International Journal on Marine Navigation and Safety of Sea Transportation
Volume 9
Number 1
March 2015

DOI: 10.12716/1001.09.01.10

Positive Descriptor Time-varying Discrete-time Linear Systems and Their Asymptotic Stability

T. Kaczorek
Bialystok University of Technology, Bialystok, Poland

ABSTRACT: The positivity and asymptotic stability of the descriptor time-varying discrete-time linear systems are addressed. The Weierstrass-Kronecker theorem on the decomposition of the regular pencil is extended to the time-varying discrete-time descriptor linear systems. Using the extension necessary and sufficient conditions for the positivity of the systems are established. Sufficient conditions for asymptotic stability of the positive systems are presented. The effectiveness of the tests is demonstrated on the example.

1 INTRODUCTION

A dynamical system is called positive if its trajectory starting from any nonnegative initial condition state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive system theory is given in the monographs Farina & Rinaldi 2000, Kaczorek 2001 and in the papers Kaczorek 1997, 1998a, 2011, 2015. Models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc.

The Laypunov, Bohl and Perron exponents and stability of time-varying discrete-time linear systems have been investigated in Czornik et. all 2012, 2013a, 2013b, 2013c, 2013d, 2014. The positive standard and descriptor systems and their stability have been analyzed in Kaczorek 1998a, 2001, 2011, 2015.The positive linear systems with different fractional orders have been addressed in Kaczorek 2011, 2012 and the singular discrete-time linear systems in Kaczorek 1998a.The switched discrete-time systems have been considered in Zhang et. all 2014a, 2014b and the extremal norms for positive linear inclusions in Zhong et. all 2013.

In this paper the positivity and asymptotic stability of the descriptor time-varying discrete-time linear systems with regular pencils will be investigated.

The paper is organized as follows. In section 2 the Weierstrass-Kronecker decomposition of the regular pencil is extended to descriptor time-varying discrete-time linear systems and the solution of the state-equation describing the time-varying discrete-time linear system is derived. Necessary and sufficient conditions for the positivity of the descriptor systems are established in section 3. The stability of the positive descriptor systems is addressed in section 4. Concluding remarks are given in section 5.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ - the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, I_n - the $n\times n$ identity matrix.

2 POSITIVE TIME-VARYING DISCRETE-TIME LINEAR SYSTEMS

Consider the descriptor time-varying discrete-time linear system

$$E(i)x_{i+1} = A(i)x_i + B(i)u_i, \; i \in Z_+ = \{0,1,...\} \quad (2.1a)$$

$$y_i = C(i)x_i \quad (2.1b)$$

where $x_i \in \Re^n$, $u_i \in \Re^m$, $y_i \in \Re^p$ are the state, input and output vectors and $A(i) \in \Re^{n \times n}$, $B(i) \in \Re^{n \times m}$, $C(i) \in \Re^{p \times n}$ are matrices with entries depending on $i \in Z_+$.

It is assumed that $\det E(i) = 0$, $i \in Z_+$ and

$$\det[E(i)\lambda - A(i)] \neq 0 \quad (2.2)$$

for some $\lambda \in C$ (the field of complex numbers) and $i \in Z_+$.

It is well-known (Kaczorek 2015) that if (2.2) holds then there exists a pair of nonsingular matrices $P(i), Q(i) \in \Re^{n \times n}$ such that

$$P(i)[E(i)\lambda - A(i)]Q(i) = \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}\lambda - \begin{bmatrix} A_1(i) & 0 \\ 0 & I_{n_2} \end{bmatrix}, \quad (2.3)$$

where $i \in Z_+$, $n_1 = \deg\det[E(i)\lambda - A(i)]$, $A_1(i) \in \Re^{n_1 \times n_1}$, $N \in \Re^{n_2 \times n_2}$ is the nilpotent matrix with the index μ (i.e. $N^\mu = 0$ and $N^{\mu-1} \neq 0$).

The matrices $P(i), Q(i), A_1(i)$ can be found by for example the use of elementary row and column operations (Kaczorek 1998b).

Premultiplying (2.1a) by the matrix $P(i)$, introducing the new state vector

$$\bar{x}_i = Q^{-1}(i)x_i = \begin{bmatrix} \bar{x}_{1,i} \\ \bar{x}_{2,i} \end{bmatrix}, \; \bar{x}_{1i} = \begin{bmatrix} \bar{x}_{11,i} \\ \bar{x}_{12,i} \\ \vdots \\ \bar{x}_{1n_1,i} \end{bmatrix}, \; \bar{x}_{2i} = \begin{bmatrix} \bar{x}_{21,i} \\ \bar{x}_{22,i} \\ \vdots \\ \bar{x}_{2n_2,i} \end{bmatrix} \quad (2.4)$$

and using (2.3) we obtain

$$\bar{x}_{1,i+1} = A_1(i)\bar{x}_{1,i} + B_1(i)u_i \quad (2.5a)$$

$$N\bar{x}_{2,i+1} = \bar{x}_{2,i} + B_2(i)u_i \quad (2.5b)$$

where

$$P(i)B(i) = \begin{bmatrix} B_1(i) \\ B_2(i) \end{bmatrix}, \; B_1(i) \in \Re^{n_1 \times m}, \; B_2(i) \in \Re^{n_2 \times m}. \quad (2.5c)$$

Theorem 2.1. The solution of equation (2.5a) for known initial condition $\bar{x}_{10} \in \Re^{n_1}$ and input $u_i \in \Re^m$, $i \in Z_+$ is given by

$$\bar{x}_{1,i} = \Phi_1(i,0)\bar{x}_{1,0} + \sum_{j=0}^{i-1}\Phi_1(i,j+1)B_1(j)u_j, \; i \in Z_+ \quad (2.6a)$$

where

$$\Phi_1(k,j) = \begin{cases} I_n & \text{for } k = j \geq 0 \\ A_1(k-1)A_1(k-2)...A_1(j) & \text{for } k > j \geq 0 \end{cases}. \quad (2.6b)$$

Proof is given in (Kaczorek 2015).

To simplify the notation it is assumed that the matrix N in (2.5b) has the form

$$N = \begin{bmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & 1 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix} \in \Re^{n_2 \times n_2}. \quad (2.7)$$

From (2.5b) and (2.7) we have

$$\begin{bmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & 1 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix}\begin{bmatrix} \bar{x}_{21,i+1} \\ \bar{x}_{22,i+1} \\ \vdots \\ \bar{x}_{2n_2,i+1} \end{bmatrix} = \begin{bmatrix} \bar{x}_{21,i} \\ \bar{x}_{22,i} \\ \vdots \\ \bar{x}_{2n_2,i} \end{bmatrix} + \begin{bmatrix} B_{21}(i) \\ \vdots \\ B_{2n_2}(i) \end{bmatrix}u_i \quad (2.8a)$$

for $i \in Z_+$ and

$$\begin{aligned} 0 &= \bar{x}_{2n_2,i} + B_{2n_2}(i)u_i, \\ \bar{x}_{2n_2,i+1} &= \bar{x}_{2n_2-1,i} + B_{2n_2-1}(i)u_i, \quad i \in Z_+. \\ &\vdots \\ \bar{x}_{22,i+1} &= \bar{x}_{21,i} + B_{21}(i)u_i \end{aligned} \quad (2.8b)$$

Solving the equations (2.8b) with respect to the components of the vector $\bar{x}_{2,i}$ we obtain

$$\begin{aligned} \bar{x}_{2n_2,i} &= -B_{2n_2}(i)u_i, \\ \bar{x}_{2n_2-1,i} &= -B_{2n_2}(i+1)u_{i+1} - B_{2n_2-1}(i)u_i, \\ &\vdots \\ \bar{x}_{21,i} &= -B_{2n_2}(i+n_2-1)u_{i+n_2-1} - ... - B_{21}(i)u_i. \end{aligned} \quad (2.9)$$

The considerations can be easily extended to the case when the matrix N in (2.5b) has the form

$$N = \text{blockdiag}[N_1, ..., N_q], \quad q > 1 \quad (2.10)$$

and N_k for $k = 1,2,…,q$ has the form (2.7).

Example 2.1. Consider the descriptor time-varying system described by the equation (2.1a) with the matrices

$$E(i)=\begin{bmatrix} 0 & 0 & 0 & \frac{e^{2i}}{\cos(i)+2} \\ \frac{(i+2)(\sin(i)+1)}{i+1} & e^{i} & 0 & -\frac{e^{2i}(e^{-i}+1)}{\cos(i)+2} \\ \frac{i+2}{i+1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B(i)=\begin{bmatrix} b_1(i) & 0 \\ b_2(i) & b_3(i) \\ 0 & b_4(i) \\ 0 & b_5(i) \end{bmatrix},$$

$$b_1(i)=\frac{1}{\cos(i)+2} \tag{2.11}$$

$$b_2(i)=e^{-i}-\frac{e^{-i}+1}{\cos(i)+2}$$

$$b_3(i)=\frac{2i(i+2)(\cos(i)+1)(\sin(i)+1)}{i+1}-\sin(i)(\sin(i)+1)$$

$$b_4(i)=\sin(i)-\frac{2i(i+2)(\cos(i)+1)}{i+1}$$

$$b_5(i)=\frac{2i(i+2)}{i+1}$$

$$A(i)=\begin{bmatrix} 0 & 0 & a_{13}(i) & 0 \\ a_{21}(i) & a_{22}(i) & a_{23}(i) & a_{24}(i) \\ a_{31}(i) & 0 & 0 & a_{34}(i) \\ 0 & 0 & 0 & a_{44}(i) \end{bmatrix},$$

where

$$a_{13}(i)=\frac{1}{\cos(i)+2},$$

$$a_{21}(i)=\frac{(i+2)(i+2\cos(i)+2\sin(i)+i\sin(i)+\cos(i)\sin(i)+3)}{(i+1)(\sin(i)+2)},$$

$$a_{22}(i)=e^{-2i}-2e^{i},\quad a_{23}(i)=-\frac{e^{-i}+1}{\cos(i)+2},$$

$$a_{24}(i)=\frac{e^{2i}(i+2)(\cos(i)+1)(\sin(i)+1)}{i+1},$$

$$a_{31}(i)=-\frac{i+2}{\sin(i)+2},\quad a_{34}(i)=-\frac{e^{2i}(i+2)(\cos(i)+1)}{i+1},$$

$$a_{44}(i)=\frac{e^{2i}(i+2)}{i+1}.$$

The condition (2.2) is satisfied since

$$\det[E(i)\lambda-A(i)] = -\frac{(i+2)^2(2e^{i}+\lambda e^{i}-1)(2\lambda+i+\lambda\sin(i)+1)}{(i+1)^2(\cos(i)+2)(\sin(i)+2)}\neq 0 \tag{2.12}$$

In this case

$$P(i)=\begin{bmatrix} 1+e^{-i} & 1 & 1+\sin(i) & 0 \\ 0 & 0 & 1 & 1+\cos(i) \\ 2+\cos(i) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{i+1}{i+2} \end{bmatrix},$$

$$Q(i)=\begin{bmatrix} 0 & \frac{i+1}{i+2} & 0 & 0 \\ e^{-i} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-2i} \end{bmatrix} \tag{2.13}$$

and

$$\begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}=P(i)E(i)Q(i)=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} A_1(i) & 0 \\ 0 & I_{n_2} \end{bmatrix}=P(i)A(i)Q(i) =\begin{bmatrix} e^{-i}-2 & 1+\cos(i) & 0 & 0 \\ 0 & -\frac{i+1}{2+\sin(i)} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} B_1(i) \\ B_2(i) \end{bmatrix}=P(i)B(i)=\begin{bmatrix} e^{-i} & 0 \\ 0 & \sin(i) \\ 1 & 0 \\ 0 & 2i \end{bmatrix}, \tag{2.14}$$

$(n_1=n_2=2)$

The equation (2.5) have the form

$$\begin{bmatrix} \bar{x}_{11,i+1} \\ \bar{x}_{12,i+1} \end{bmatrix}=\begin{bmatrix} e^{-i}-2 & 1+\cos(i) \\ 0 & -\frac{i+1}{2+\sin(i)} \end{bmatrix}\begin{bmatrix} x_{11,i} \\ x_{12,i} \end{bmatrix}+\begin{bmatrix} e^{-i} & 0 \\ 0 & \sin(i) \end{bmatrix}\begin{bmatrix} u_{1i} \\ u_{2i} \end{bmatrix} \tag{2.15a}$$

and

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \bar{x}_{21,i+1} \\ \bar{x}_{22,i+1} \end{bmatrix}=\begin{bmatrix} \bar{x}_{21,i} \\ \bar{x}_{22,i} \end{bmatrix}+\begin{bmatrix} 1 & 0 \\ 0 & 2i \end{bmatrix}\begin{bmatrix} u_{1i} \\ u_{2i} \end{bmatrix} \tag{2.15b}$$

The solution of (2.15a) is given by (2.6) with the matrices $A_1(i)$ and $B_1(i)$ defined by (2.14).

From (2.15b) we have

$$\begin{aligned} \bar{x}_{22,i} &= -2iu_{2i}, \\ \bar{x}_{21,i} &= -u_{1i}+\bar{x}_{21,i+1}=-u_{1i}-2(i+1)u_{2i+1},\quad i\in Z_+. \end{aligned} \tag{2.16}$$

The solution of the equation (2.1a) with (2.11) is given by

$$x(i)=\begin{bmatrix}x_1(i)\\x_2(i)\\x_3(i)\\x_4(i)\end{bmatrix}=Q(i)\begin{bmatrix}\bar{x}_{11,i}\\\bar{x}_{12,i}\\\bar{x}_{21,i}\\\bar{x}_{22,i}\end{bmatrix},\quad i\in Z_+ \tag{2.17}$$

where $Q(i)$ is defined by (2.13) and the components of the state vector $\bar{x}(i)$ by (2.6) with $A_1(i)$ and $B_1(i)$ defined by (2.14) and (2.16).

3 POSITIVE SYSTEMS

Definition 3.1. The descriptor time-varying discrete-time linear system (2.1) is called the (internally) positive if and only if $x_i\in\Re_+^n$ and $y_i\in\Re_+^p$, $i\in Z_+$ for any admissible initial conditions $x_0\in\Re_+^n$ and all inputs $u_i\in\Re_+^m$, $i\in Z_+$.

The matrix $Q(i)\in\Re^{n\times n}$, $i\in Z_+$ is called monomial if in each row and column only one entry is positive and the remaining entries are zero for all $i\in Z_+$.

It is well-known (Kaczorek 1998a) that $Q^{-1}(i)\in\Re_+^{n\times n}$, $i\in Z_+$ if and only if the matrix is monomial.

It is assumed that for the positive system (2.1) the decomposition (2.3) is positive for the monomial matrix $Q(i)$. In this case

$$x_i=Q(i)\bar{x}_i\in\Re_+^n \text{ if and only if } \bar{x}_i\in\Re_+^n,\ i\in Z_+. \tag{3.1}$$

It is also well-known that premultiplication of the equation (2.1a) by the matrix $P(i)$ does not change its solution x_i, $i\in Z_+$.

From (2.9) it follows that $\bar{x}_{2,i}\in\Re_+^{n_2}$, $i\in Z_+$ for $u_i\in\Re_+^m$, $i\in Z_+$ if and only if

$$-B_2(i)\in\Re_+^{n_2\times m} \text{ for } i\in Z_+. \tag{3.2}$$

In (Kaczorek 2015) has been shown that the time-varying discrete-time system (2.5a) is positive if and only if

$$A_1(i)\in\Re_+^{n_1\times n_1},\quad B_1(i)\in\Re_+^{n_1\times m},\ i\in Z_+. \tag{3.3}$$

From (2.1b) and (2.4) we have

$$y_i=C(i)Q(i)Q^{-1}(i)x_i=\bar{C}(i)\bar{x}_i,\ i\in Z_+. \tag{3.4a}$$

where

$$\bar{C}(i)=C(i)Q(i). \tag{3.4b}$$

For monomial matrix $Q(i)\in\Re_+^{n\times n}$ from (3.4) we have $\bar{C}(i)\in\Re_+^{p\times n}$, $i\in Z_+$ if and only if $C(i)\in\Re_+^{p\times n}$, $i\in Z_+$ and $y_i\in\Re_+^p$, $i\in Z_+$.

Therefore, the following theorem has been proved.

Theorem 3.1. The descriptor time-varying discrete-time linear system (2.1) is positive if and only if

1 there exists the decomposition (2.3) for monomial matrix $Q(i)\in\Re_+^{n\times n}$, $i\in Z_+$;
2 the conditions (3.2) and (3.3) are satisfied;
3 $C(i)\in\Re_+^{p\times n}$ for $i\in Z_+$.

Example 3.1. Consider the descriptor time-varying system described by the equation (2.1) with the matrices

$$E(i)=\begin{bmatrix}0&0&0&\frac{1}{2\sin(i)+4}\\-\cos(i)-1&\frac{1}{\cos(i)+2}&0&-\frac{e^{-i}+2}{2\sin(i)+4}\\1&0&0&0\\0&0&0&0\end{bmatrix},$$

$$B(i)=\begin{bmatrix}-\frac{1}{\sin(i)+2}&0\\e^{-i}+\frac{e^{-i}+2}{\sin(i)+2}&-(\cos(i)+1)(e^{-i}+\sin(i)+2)\\0&e^{-i}+\sin(i)+2\\0&-1\end{bmatrix},$$

$$C(i)=\begin{bmatrix}0&\frac{1}{\cos(i)+2}&0&0.5\\\frac{i+2}{i+1}&0&\frac{e^{-i}}{e^{-i}+1}&0\end{bmatrix}, \tag{3.5}$$

$$A(i)=\begin{bmatrix}0&0&a_{13}(i)&0\\a_{21}(i)&a_{22}(i)&a_{23}(i)&a_{24}(i)\\a_{31}(i)&0&0&a_{34}(i)\\0&0&0&a_{44}(i)\end{bmatrix},$$

where

$$a_{13}(i)=\frac{1}{(\sin(i)+2)(e^{-i}+1)},$$

$$a_{21}(i)=-0.3e^{-i}-0.3\cos(i)-0.2\sin(i)-0.3e^{-i}\cos(i)-0.1,$$

$$a_{22}(i)=\frac{0.1(i+1)}{(i+2)(\cos(i)+2)},$$

$$a_{23}(i)=-\frac{e^{-i}+2}{(\sin(i)+2)(e^{-i}+1)},$$

$$a_{24}(i)=\frac{(i+2)(\cos(i)+1)(e^{-i}+1)}{2(i+1)},\ a_{31}(i)=0.3(e^{-i}+1),$$

$$a_{34}(i)=-\frac{(i+2)(e^{-i}+1)}{2(i+1)},\quad a_{44}(i)=\frac{i+2}{2(i+1)}.$$

The condition (2.2) is satisfied since

$$\det[E(i)\lambda - A(i)] = \frac{e^{-i}(3i-60\lambda-30i\lambda+3)+3i-70\lambda+100i\lambda^2+200\lambda^2-40i\lambda+3}{100(i+1)(e^{-i}+1)(\sin(2i)+4\cos(i)+4\sin(i)+8)} \neq 0 \quad (3.6)$$

In this case

$$P = \begin{bmatrix} 2+e^{-i} & 1 & 1+\cos(i) & 0 \\ 0 & 0 & 1 & 1+e^{-i} \\ 2+\sin(i) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{i+1}{i+2} \end{bmatrix},$$

$$Q = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2+\cos(i) & 0 & 0 & 0 \\ 0 & 0 & 1+e^{-i} & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad (3.7)$$

and

$$\begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix} = P(i)E(i)Q(i) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} A_1(i) & 0 \\ 0 & I_{n_2} \end{bmatrix} = P(i)A(i)Q(i) \quad (3.8)$$

$$= \begin{bmatrix} 0.1\frac{i+1}{i+2} & 0.2(1-\sin(i)) & 0 & 0 \\ 0 & 0.3(1+e^{-i}) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} B_1(i) \\ B_2(i) \end{bmatrix} = P(i)B(i) = \begin{bmatrix} e^{-i} & 0 \\ 0 & 1+\sin(i) \\ -1 & 0 \\ 0 & -\frac{i+1}{i+2} \end{bmatrix}, \quad (3.8)$$

$$\bar{C}(i) = C(i)Q(i) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & \frac{i+2}{i+1} & e^{-i} & 0 \end{bmatrix}.$$

The descriptor system is positive since the three conditions of Theorem 3.1 are satisfied. The matrix $Q(i)$ defined by (3.7) is monomial, the conditions (3.2) and (3.3) are met since

$$-B_2(i) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{i+1}{i+2} \end{bmatrix} \in R_+^{2\times 2},$$

$$A_1(i) = \begin{bmatrix} 0.1\frac{i+1}{i+2} & 0.2(1-\sin(i)) \\ 0 & 0.3(1+e^{-i}) \end{bmatrix} \in R_+^{2\times 2}, \quad (3.9)$$

$$B_1(i) = \begin{bmatrix} e^{-i} & 0 \\ 0 & 1+\sin(i) \end{bmatrix} \in R_+^{2\times 2}, \quad i \in Z_+$$

and

$$C(i) = \begin{bmatrix} 0 & \frac{1}{\cos(i)+2} & 0 & 0.5 \\ \frac{i+2}{i+1} & 0 & \frac{e^{-i}}{e^{-i}+1} & 0 \end{bmatrix} \in R_+^{2\times 4} \text{ for } Z_+.$$

The solution to the equation (2.1) with the matrices $E(i)$, $A(i)$, $B(i)$ given by (3.5) can be found in a similar way as in Example 2.1.

4 STABILITY OF THE POSITIVE DESCRIPTOR LINEAR SYSTEMS

From (2.1a) and (2.6a) for $E(i) = I_n$, $B(i)u_i = 0$, $i \in Z_+$ it follows that

$$\hat{x}_{1,i} = \Phi_1(i)\bar{x}_{1,0}, \quad i \in Z_+ \quad (4.1a)$$

where

$$\Phi_1(i) = \Phi_1(i,0) = A_1(i-1)A_1(i-2)...A_1(0) \quad (4.1b)$$

is the solution of the equation

$$\bar{x}_{1,i+1} = A_1(i)\bar{x}_{1,i}, \quad i \in Z_+. \quad (4.2)$$

From (4.1b) we have

$$\Phi_1(i+1) = A_1(i)\Phi_1(i), \quad i \in Z_+. \quad (4.3)$$

Definition 4.1. The positive system (4.2) is called asymptotically stable if the norm $\|\bar{x}_{1,i}\|$ of the state vector $\bar{x}_{1,i} \in \Re_+^{n_1}$, $i \in Z_+$ satisfies the condition

$$\lim_{i\to\infty} \|\bar{x}_{1,i}\| = 0 \text{ for any finite } \bar{x}_{10} \in \Re_+^{n_1}. \quad (4.4)$$

Theorem 4.1. The positive system (4.2) is asymptotically stable if the norm $\|A_1(i)\|$ of the matrix $A\,(i)$, $i \in Z_+$ satisfies the condition

$$\|A_1\| < 1 \text{ for } i \in Z_+ \quad (4.5a)$$

where

$$\|A_1\| \geq \max_{0\leq i\leq\infty} \|A_1(i)\| \text{ for } i \in Z_+. \quad (4.5b)$$

Proof is given in (Kaczorek 2015).

Theorem 4.2. The positive system (4.2) is asymptotically stable if its system matrix $A_1(i) = [a_{jk}^{(1)}(i)] \in \Re_+^{n\times n}$ satisfies the condition

$$\max_{0\le j\le n_1}\sum_{k=1}^{n_1} a_{jk}^{(1)}(i) < 1 \quad \text{for} \quad i \in Z_+ \tag{4.6a}$$

or

$$\max_{0\le k\le n_1}\sum_{j=1}^{n_1} a_{jk}^{(1)}(i) < 1 \quad \text{for} \quad i \in Z_+ . \tag{4.6b}$$

Proof is given in (Kaczorek 2015).

Theorem 4.3. The positive system (4.2) is asymptotically stable if its system matrix

$$A_1(i) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ a_0^{(1)}(i) & a_1^{(1)}(i) & a_2^{(1)}(i) & \dots & a_{n-1}^{(1)}(i) \end{bmatrix} \in \Re_+^{n_1\times n_1} \tag{4.7}$$

satisfies the condition

$$\sum_{k=0}^{n-1} a_k^{(1)}(i) < 1 \quad \text{for} \quad i \in Z_+ . \tag{4.8}$$

Proof is given in (Kaczorek 2015).

Consider the positive descriptor system described by (2.1a) for $B(i)u_i = 0$, $i \in Z_+$

$$E(i)x_{i+1} = A(i)x_i . \tag{4.9}$$

If the assumption (2.2) is satisfied then the characteristic polynomial of the system (4.9) and of the system

$$\begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix} \bar{x}_{i+1} = \begin{bmatrix} A_1(i) & 0 \\ 0 & I_{n_2} \end{bmatrix} \bar{x}_i \tag{4.10}$$

are related by

$$\begin{aligned} p(z,i) &= \det[E(i)z - A(i)] \\ &= \det\left[P^{-1}(i)\left[\begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix} z - \begin{bmatrix} A_1(i) & 0 \\ 0 & I_{n_2} \end{bmatrix}\right] Q^{-1}(i)\right] \\ &= k(i)\bar{p}(z,i) \end{aligned} \tag{4.11a}$$

where

$$\begin{aligned} \bar{p}(z,i) &= \det\left[\begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix} z - \begin{bmatrix} A_1(i) & 0 \\ 0 & I_{n_2} \end{bmatrix}\right] \\ &= \det[I_{n_1} z - A_1(i)], \\ k(i) &= \det P^{-1}(i) Q^{-1}(i). \end{aligned} \tag{4.11b}$$

From (4.11) we have the following lemma.

Lemma 4.1. The positive descriptor time-varying discrete-time linear system (4.9) is asymptotically stable if and only if the positive time-varying linear system

$$\bar{x}_{1,i+1} = A_1(i)\bar{x}_{1,i} \tag{4.12}$$

is asymptotically stable.

From Theorem 4.1 and Lemma 4.1 we have the following theorem.

Theorem 4.4. The positive descriptor time-varying discrete-time linear system (4.9) is asymptotically stable if the condition

$$\|A_1\| = \max_{0\le i\le\infty}\|A_1(i)\| < 1 \tag{4.13}$$

is satisfied.

Similarly, from Theorem 4.2, 4.3 and Lemma 4.1 we have the following theorems.

Theorem 4.5. The positive descriptor time-varying discrete-time linear system (4.9) is asymptotically stable if the matrix $A_1(i) = [a_{jk}^{(1)}(i)] \in \Re_+^{n_1\times n_1}$, $i \in Z_+$ satisfies the condition

$$\max_{0\le j\le n}\sum_{k=1}^{n} a_{jk}^{(1)}(i) < 1 \quad \text{for} \quad i \in Z_+ \tag{4.14a}$$

or

$$\max_{0\le k\le n}\sum_{j=1}^{n} a_{jk}^{(1)}(i) < 1 \quad \text{for} \quad i \in Z_+ . \tag{4.14b}$$

Theorem 4.6. The positive descriptor time-varying discrete-time linear system (4.9) is asymptotically stable if the matrix $A_1(i)$ has the canonical Frobenius form

$$A_1(i) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ a_0^{(1)}(i) & a_1^{(1)}(i) & a_2^{(1)}(i) & \dots & a_{n-1}^{(1)}(i) \end{bmatrix} \in \Re_+^{n_1\times n_1} \tag{4.15}$$

and it satisfies the condition

$$\sum_{k=0}^{n-1} a_k^{(1)}(i) < 1 \quad \text{for} \quad i \in Z_+ . \tag{4.16}$$

Example 4.1. (continuation of Example 3.1). By Theorem 4.4 the positive descriptor time-varying discrete-time linear system (2.1) with the matrix *A*(*i*) given by (3.5) is asymptotically stable since

$$\|A_1\| = \max_{0\le i\le\infty}\|A_1(i)\| = \max_{0\le i\le\infty}\left\{0.1\frac{i+1}{i+2}+0.2[1-\sin(i)],\ \ 0.3(1-e^{-i})\right\}<1 \tag{4.17}$$

for all $i \in Z_+$.

5 CONCLUDING REMARKS

The positivity and asymptotic stability of the descriptor time-varying discrete-time linear systems with regular pencils have been addressed. The Weierstrass-Kronecker theorem on the decomposition of the regular pencils has been extended to the descriptor time-varying discrete-time linear systems. Solutions to the decomposed systems have been derived (Theorem 2.1). Necessary and sufficient conditions for the positivity of the systems have been established (Theorem 3.1). Using the norms of the vectors and matrices sufficient conditions for asymptotic stability of the positive systems have been derived (Theorems 4.1 – 4.6). The effectiveness of the test are demonstrated on examples. The proposed method can be applied in analysis of marine navigation and safety of sea transportation problems. The considerations can be extended to the fractional descriptor time-varying discrete-time linear systems.

ACKNOWLEDGMENT

This work was supported by National Science Centre in Poland under work No. 2014/13/B/ST7/03467.

REFERENCES

Czornik A., Newrat A., Niezabitowski M., Szyda A. 2012. On the Lyapunov and Bohl exponent of time-varying discrete linear systems, 20th Mediterranean Conf. on Control and Automation (MED), Barcelona, 194-197.

Czornik A., Niezabitowski M. 2013a. Lyapunov Exponents for Systems with Unbounded Coefficients, Dynamical Systems: an International Journal, vol. 28, no. 2, 140-153.

Czornik A., Newrat A., Niezabitowski M. 2013, On the Lyapunov exponents of a class of the secon order discrete time linear systems with bounded perturbations, Dynamical Systems: an International Journal, vol. 28, no. 4, 473-483.

Czornik A., Niezabitowski M. 2013b. On the stability of discrete time-varying linear systems, Nonlinear Analysis: Hybrid Systems. vol. 9, 27-41.

Czornik A., Niezabitowski M. 2013c. On the stability of Lyapunov exponents of discrete linear system, Proc. of European Control Conf., Zurich, 2210-2213.

Czornik A., Klamka J., Niezabitowski M. 2014. On the set of Perron exponents of discrete linear systems, Proc. of World Congres of the 19th International Federation of Automatic Control, Kapsztad, 11740-11742.

Farina L., Rinaldi S. 2000. Positive Linear Systems; Theory and Applications, J. Wiley, New York.

Kaczorek T. 2001. Positive 1D and 2D systems, Springer Verlag, London.

Kaczorek T. 2011. Positive linear systems consisting of n subsystems with different fractional orders, IEEE Trans. Circuits and Systems, vol. 58, no. 6, 1203-1210.

Kaczorek T. 1998a. Positive descriptor discrete-time linear systems, Problems of Nonlinear Analysis in Engineering Systems, vol. 1, no. 7, 38-54.

Kaczorek T. 2015. Positive descriptor time-varying discrete-time linear systems, Proc. of Conf. ACIIDS, Bali, Indonesia, Springer-Verlag.

Kaczorek T. 1997. Positive singular discrete time linear systems, Bull. Pol. Acad. Techn. Sci., vol. 45, no 4, 619-631.

Kaczorek T. 2012. Selected Problems of Fractional Systems Theory, Springer-Verlag, Berlin.

Kaczorek T. 1998b. Vectors and Matrices in Automation and Electrotechics, WNT, Warszawa (in Polish).

Rami M. A., Bokharaie V.S., Mason O., Wirth F.R. 2012. Extremel norms for positive linear inclusions, 20th International Symposium on Mathematical Theory of Networks and Systems, Melbourne.

Zhang H., Xie D., Zhang H., Wang G. 2014. Stability analysis for discrete-time switched systems with unstable subsytems by a mode-dependent average dwell time approach, ISA Transactions, vol. 53, 1081-1086.

Zhang J., Han Z., Wu H., Hung J. 2014. Robust stabilization of discrete-time positive switched systems with uncertainties and average dwell time switching, Circuits Syst, Signal Process., vol. 33, 71-95.

Zhong Q., Cheng J., Zhong S. 2013. Finite-time H∞ control of a switched discrete-time system with average dwell time, Advances in Difference Equations, vol. 191.