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Kalman-Bucy Filter Design for Multivariable Ship Motion Control

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ABSTRACT: The paper presents a concept of Kalman-Bucy filter which can be used in the multivariable ship motion control system. The navigational system usually measures ship position coordinates and the ship heading, while the velocities are to be estimated using an available mathematical model of the ship. The designed Kalman-Bucy filter has been simulated on a computer model and implemented on the training ship to demonstrate the filtering properties.

1 INTRODUCTION

Modern ships are equipped with complicated ship motion control systems, the goals of which depend on tasks realised by an individual ship. The tasks executed by the control system include, among other actions, controlling the ship motion along the course or a given trajectory (path following and trajectory tracking), dynamical positioning and reduction of ship rolls caused by waves. Figure 1 presents basic components of the ship motion control system.

The guidance system generates a required smooth reference trajectory, described using given positions, velocities, and accelerations. The trajectory is generated by algorithms which make use of the required and current ship positions, and the mathematical model with complementary information on the executed task and, possibly, the weather.

The control system processes the motion related signals and generates the set values for actuators to reduce the difference between the desired trajectory and the current trajectory. The controller can have a number of operating modes depending on the executed tasks. On some ships and in some operations the required control action can be executed in several ways due to the presence of a number of propellers. Different combinations of actuators can generate the same control action. In those cases the control system has also to solve the control allocation problem, based on the optimisation criteria (Fossen, 2002).

The navigation system measures the ship position and the heading angle, collects data from various sensors, such as GPS, log, compass, gyro-compass, radar. The navigation system also checks the quality of the signal, passes it to the observer system in which the disturbances are filtered out and the ship state variables are calculated. Stochastic nature of the forces generated by the environment requires the use of observers for estimating variables related with the moving ship and for filtering the disturbances in order to use the signals in the ship motion control systems.

Filtering and estimating are extremely important properties in the multivariable control systems. In many cases the ship velocity measurements are not directly available, and the velocity estimates are to be calculated from the position and heading values measured by the observer. Unfortunately, these measurements are burdened with errors generated by environmental disturbances like wind, sea currents and waves, as well as by sensor noise.

One year after publishing his work on a discrete filter (Kalman, 1960), Rudolph Kalman, this time together with Richard Bucy, published the second work in which they discussed the problem of continuous filtering (Kalman & Bucy, 1961). This work has also become the milestone in the field of optimal filtering. In the present article the continuous Kalman filter is derived based on the discrete Kalman filter, assuming that the sampling time tends to zero. A usual tendency in numerical calculations is rather reverse: starting from continuous dynamic equations, which are digitised to arrive at the discrete difference equations being the approximates of the initial continuous dynamics. In the Kalman filter idea the discrete equations are accurate as they base on accurate difference equations of the model of the process.

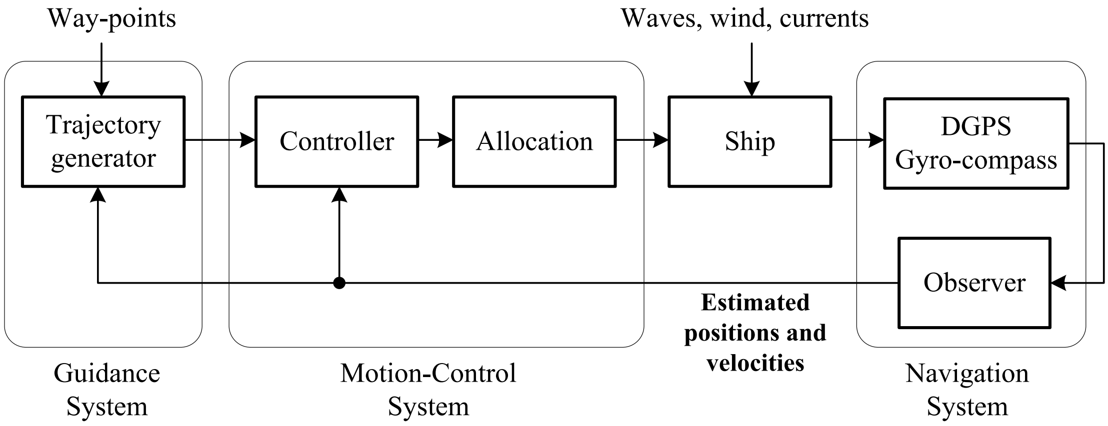

Figure 1. Basic components of modern ship motion control system (Fossen, 2002).

The dynamic positioning systems have been developed since the early sixties of the last century. The first dynamic control systems were designed using conventional PID controllers working in cascade with low-pass filters or cut-off filters to separate the motion components connected with the sea waves. However, those systems introduce phase delays which worsen the quality of the control (Fossen, 2002).

From the middle of 1970s more advanced control techniques started to be used, which were based on optimal control and the Kalman filter theory. The first solution of this type was presented by (Balchen et al., 1976). It was then modified and extended by Balchen himself and other researchers: (Balchen et al., 1980a; Balchen et al., 1980b; Fung and Grimble, 1983; Saelid et al. 1983; Sorensen et al., 1996; Strand et al. 1997). The new solutions made use of the linear theory, according to which the kinematic characteristics of the ship were to be linearized in the form of sets of predefined ship heading angles, with an usual resolution of 10 degrees. After the linearization of the nonlinear model, the observer based on such a model is only locally correct. This is the disadvantage of the Kalman filter. The Kalman filter can make use of measurements done by different sensors at different accuracy levels, and calculate ship velocity estimates which are not measured in the majority of ship positioning applications.

The main goal of the article is designing and testing the observer for the ship motion velocity estimation.

2 DISCRETE MODEL OF THE PROCESS

Discussed are time-dependent discrete processes, which are recorded by sampling continuous processes at discrete times. Let us assume that the continuous process is described by the following equation:

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{G}(t)\mathbf{u}(t) \quad (1)$$

where $\mathbf{u}$ is the input vector having the form of white noise. The state transition matrix for equation (1) takes the form:

$$\mathbf{x}(t) = e^{\mathbf{A}(t_0)(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{A}(\tau)(t-\tau)}\mathbf{G}(\tau)u(\tau)d\tau \quad (2)$$

For the discrete model, the objects of analysis are process samples recorded at times t_0, t_1, ..., t_k, ... Equation (2) written for a single sampling interval can be presented as

$$\mathbf{x}(t_{k+1}) = \mathbf{F}(t_{k+1}, t_k)\mathbf{x}(t_k) + \int_{t_k}^{t_{k+1}} \mathbf{F}(t_{k+1}, \tau)\mathbf{G}(\tau)u(\tau)d\tau \quad (3)$$

which can be briefly written as

$$\mathbf{x}_{k+1} = \mathbf{F}_k\mathbf{x}_k + \mathbf{w}_k \quad (4)$$

where $\mathbf{F}_k$ is the state transition matrix for the step between times t_k and t_{k+1} at the absence of the excitation function

$$\mathbf{F}_k = \mathbf{F}(t_{k+1}, t_k) = e^{\mathbf{A}(t_k)\cdot T} = \mathbf{I} + \mathbf{A}(t_k)\cdot T \quad (5)$$

and $\mathbf{w}_k$ is the excited response at time t_{k+1} due to the presence of the white noise at the input in the time interval (t_k, t_{k+1}), i.e. accidental disturbances affecting the process

$$\mathbf{w}_k = \int_{t_k}^{t_{k+1}} \mathbf{F}(t_{k+1}, \tau)\mathbf{G}(\tau)u(\tau)d\tau \quad (6)$$

The white noise is a stochastic signal having the mean value equal to zero and finite variance. The matrix elements $\mathbf{w}_k$ can reveal non-zero cross correlation at some times t_k. The covariance matrix connected with $\mathbf{w}_k$ is denoted as

$$E\{\mathbf{w}_k\mathbf{w}_i^T\} = \mathbf{Q}_k \quad (7)$$

The covariance matrix $\mathbf{Q}_k$ can be determined using the formula written in the following integral form

$$\mathbf{Q}_k = E\left\{\mathbf{w}_k \mathbf{w}_i^T\right\}$$
$$= E\left\{\left[\int_{t_k}^{t_{k+1}} \mathbf{F}(t_{k+1},\xi)\mathbf{G}(\xi)u(\xi)d\xi\right]\left[\int_{t_k}^{t_{k+1}} \mathbf{F}(t_{k+1},\eta)\mathbf{G}(\eta)u(\eta)d\eta\right]^T\right\}$$
$$= \int_{t_k}^{t_{k+1}}\int_{t_k}^{t_{k+1}} \mathbf{F}(t_{k+1},\xi)\mathbf{G}(\xi)E\left[\mathbf{u}(\xi)\mathbf{u}^T(\eta)\right]\mathbf{G}^T(\eta)\mathbf{F}^T(t_{k+1},\eta)d\xi d\eta \quad (8)$$

The matrix $E[\mathbf{u}(\xi)\mathbf{u}^T(\eta)]$ is the matrix of the Dirac delta function, well known from continuous models.

3 DISCRETE KALMAN FILTER

Briefly, the Kalman filter tries to estimate, in an optimal way, the state vector of the controlled process modelled by the linear and stochastic difference equation having the form given by formula (4). Observations (measurements) of the process are done at discrete times and meet the following linear relation

$$\mathbf{y}_k = \mathbf{H}_k \mathbf{x}_k + \mathbf{v}_k \quad (9)$$

where $\mathbf{x}_k$ is the state vector of the process at time t_k, $\mathbf{y}_k$ is the vector of the values measured at time t_k, $\mathbf{H}_k$ is the matrix representing the relation between the measurements and the state vector at time t_k, and $\mathbf{v}_k$ represents the measurement errors. It is assumed that the signals $\mathbf{v}_k$ and $\mathbf{w}_k$ have the mean value equal to zero and there is no correlation between them.

The covariance matrix for the vector $\mathbf{w}_k$ is given by formula (7), while that for $\mathbf{v}_k$ is defined in the following way

$$E\left\{\mathbf{v}_k \mathbf{v}_k^T\right\} = \mathbf{R}_k \quad (10)$$

$$E\left\{\mathbf{w}_k \mathbf{v}_k^T\right\} = 0, \quad \text{for all } k \text{ and } i \quad (11)$$

It is assumed that the initial values of the process estimates are known at the beginning time t_k and that these estimates until time t_k base on the knowledge about the process. Such an estimate is denoted as $\mathbf{x}_k$ where the bar means that this is the best estimate at time t_k before the measurement. The estimation error is defined as:

$$\bar{\mathbf{e}}_k = \mathbf{x}_k - \bar{\mathbf{x}}_k \quad (12)$$

and the related error covariance matrix is

$$\bar{\mathbf{P}}_k = E\left\{\bar{\mathbf{e}}_k \bar{\mathbf{e}}_k^T\right\} = E\left\{\left(\mathbf{x}_k - \bar{\mathbf{x}}_k\right)\left(\mathbf{x}_k - \bar{\mathbf{x}}_k\right)^T\right\} \quad (13)$$

At sampling times t_k at which the measurement $\mathbf{y}_k$ is done, the possessed estimate $\mathbf{x}_k$ is corrected using the following relation (Brown & Hwang, 1997; Franklin et al. 1998)

$$\hat{\mathbf{x}}_k = \bar{\mathbf{x}}_k + \mathbf{L}_k\left(\mathbf{y}_k - \mathbf{H}_k \bar{\mathbf{x}}_k\right) \quad (14)$$

where $\hat{\mathbf{x}}_k$ is the estimate updated by the performed measurement, and $\mathbf{L}_k$ is the scaling amplification. The task is to find the vector amplifications $\mathbf{L}_k$ which update the estimate in the optimal way. For this purpose the minimisation of the mean square error is done. Then, the covariance matrix is determined for the error relating to the estimate updated by the performed measurement.

$$\mathbf{P}_k = E\left\{\mathbf{e}_k \mathbf{e}_k^T\right\} = E\left\{(\mathbf{x}_k - \hat{\mathbf{x}}_k)(\mathbf{x}_k - \hat{\mathbf{x}}_k)^T\right\} \quad (15)$$

In time intervals between the sampling times, the estimates are calculated using the following formula

$$\bar{\mathbf{x}}_{k+1} = \mathbf{F}_k \hat{\mathbf{x}}_k \quad (16)$$

Firstly, the covariance matrix $\bar{\mathbf{P}}_{k+1}$ is calculated using formula (13) after correcting it by one sample ahead

$$\bar{\mathbf{P}}_{k+1} = E\left\{\left(\mathbf{x}_{k+1} - \bar{\mathbf{x}}_{k+1}\right)\left(\mathbf{x}_{k+1} - \bar{\mathbf{x}}_{k+1}\right)^T\right\} \quad (17)$$

After placing relations (4) and (16) into formula (17) we get (Brown & Hwang, 1997)

$$\bar{\mathbf{P}}_{k+1} = E\left\{\left[\mathbf{F}_k(\mathbf{x}_k - \hat{\mathbf{x}}_k) + \mathbf{w}_k\right]\left[\mathbf{F}_k(\mathbf{x}_k - \hat{\mathbf{x}}_k) + \mathbf{w}_k\right]^T\right\}$$
$$= \mathbf{F}_k E\left\{(\mathbf{x}_k - \hat{\mathbf{x}}_k)(\mathbf{x}_k - \hat{\mathbf{x}}_k)^T\right\}\mathbf{F}_k^T + \mathbf{F}_k E\left\{(\mathbf{x}_k - \hat{\mathbf{x}}_k)\mathbf{w}_k^T\right\}$$
$$+ E\left\{\mathbf{w}_k(\mathbf{x}_k - \hat{\mathbf{x}}_k)^T\right\}\mathbf{F}_k^T + E\left\{\mathbf{w}_k \mathbf{w}_k^T\right\} \quad (18)$$

No correlation is assumed between the estimation error signals $\mathbf{e}_k$ and the disturbances $\mathbf{w}_k$. After placing the covariances defined by formulas (7) and (15) into relation (18) we get the required error covariance matrix between the sampling times

$$\bar{\mathbf{P}}_{k+1} = \mathbf{F}_k \mathbf{P}_k \mathbf{F}_k^T + \mathbf{Q}_k \quad (19)$$

The estimation error covariance matrix $\mathbf{P}_k$ is calculated for sampling times in the similar way. After placing relations (9) and (14) into formula (15) we get

$$\mathbf{P}_k = E\left\{\left[\left(\mathbf{x}_k - \bar{\mathbf{x}}_k\right) - \mathbf{L}_k\mathbf{H}_k\left(\mathbf{x}_k - \bar{\mathbf{x}}_k\right) - \mathbf{L}_k\mathbf{v}_k\right]\right.$$
$$\left.\cdot\left[\left(\mathbf{x}_k - \bar{\mathbf{x}}_k\right) - \mathbf{L}_k\mathbf{H}_k\left(\mathbf{x}_k - \bar{\mathbf{x}}_k\right) - \mathbf{L}_k\mathbf{v}_k\right]^T\right\} \quad (20)$$

And after similar algebra operations as in formula (21) we get the following solution

$$\mathbf{P}_k = \bar{\mathbf{P}}_k - \mathbf{L}_k\mathbf{H}_k\bar{\mathbf{P}}_k - \bar{\mathbf{P}}_k\mathbf{H}_k^T\mathbf{L}_k^T$$
$$+ \mathbf{L}_k\left(\mathbf{H}_k\bar{\mathbf{P}}_k\mathbf{H}_k^T + \mathbf{R}_k\right)\mathbf{L}_k^T \quad (21)$$

The final task is to calculate the optimal values of the amplification matrix $\mathbf{L}_k$. This task is realised by finding such values of the vector $\mathbf{L}_k$. which minimise the trace of the matrix $\mathbf{P}_k$ being the sum of the mean square errors of the estimates of all state vector elements. The trace of the matrix $\mathbf{P}_k$ is differentiated with respect to $\mathbf{L}_k$ and made equal to zero. What can be easily noticed, the second and third term of equation (21) are linear with respect to $\mathbf{L}_k$, while the fourth term is quadratic and the trace of $\mathbf{L}_k\mathbf{H}_k\overline{\mathbf{P}}_k$ is equal to the trace of its transposition $\mathbf{L}_k\mathbf{H}_k\overline{\mathbf{P}}_k$ $\overline{\mathbf{P}}_k\mathbf{H}_k^T\mathbf{L}_k^T$. We get (Brown & Hwang, 1997)

$$\frac{d\left(\text{trace}\mathbf{P}_k\right)}{d\mathbf{L}_k} = -2\left(\mathbf{H}_k\overline{\mathbf{P}}_k^T\right)^T + 2\mathbf{L}_k\left(\mathbf{H}_k\overline{\mathbf{P}}_k\mathbf{H}_k^T + \mathbf{R}_k\right) = 0 \tag{22}$$

and after some transformations we arrive at the required form of the matrix $\mathbf{L}_k$, bearing the name of Kalman amplification:

$$\mathbf{L}_k = \overline{\mathbf{P}}_k\mathbf{H}_k^T\left(\mathbf{H}_k\overline{\mathbf{P}}_k\mathbf{H}_k^T + \mathbf{R}_k\right)^{-1} \tag{23}$$

Now we remove the matrix $\mathbf{L}_k$ from equation (21) by placing the relation (23) to get

$$\mathbf{P}_k = \overline{\mathbf{P}}_k - \overline{\mathbf{P}}_k\mathbf{H}_k^T\left(\mathbf{H}_k\overline{\mathbf{P}}_k\mathbf{H}_k^T + \mathbf{R}_k\right)^{-1}\mathbf{H}_k\overline{\mathbf{P}}_k \tag{24}$$

The obtained recurrent calculation algorithm, based on equations (14), (16), (19) (23) and (24), is widely known as the Kalman filter. Equation (24) can be presented in another form taking into account the $\mathbf{L}_k$ relation given by the formula (23) which gives

$$\mathbf{P}_k = \overline{\mathbf{P}}_k - \mathbf{L}_k\mathbf{H}_k\overline{\mathbf{P}}_k = \left(\mathbf{I} - \mathbf{L}_k\mathbf{H}_k\right)\overline{\mathbf{P}}_k \tag{25}$$

4 CONVERSION FROM DISCRETE TO CONTINUOUS EQUATIONS DESCRIBING THE KALMAN FILTER ALGORITHM

The general form of a continuous process is already given by equation (1), while the equation describing the measuring model for a continuous system is

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{v} \tag{26}$$

Consequently, by analogy with the discrete model it is assumed that the vectors $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are the vectors of the stochastic process bearing the name of the white noise with the zero cross correlation value.

$$E\left\{\mathbf{u}(t)\mathbf{u}^T(\tau)\right\} = \mathbf{Q}\delta(t-\tau) \tag{27}$$

$$E\left\{\mathbf{v}(t)\mathbf{v}^T(\tau)\right\} = \mathbf{R}\delta(t-\tau) \tag{28}$$

$$E\left\{\mathbf{u}(t)\mathbf{v}^T(\tau)\right\} = 0 \tag{29}$$

The covariance parameters $\mathbf{Q}$ and $\mathbf{R}$ play a similar role to that played by the parameters $\mathbf{Q}_k$ (7) and $\mathbf{R}_k$ (10) in the discrete filter, but have different numerical values.

To do the conversion from discrete to continuous form, let us first define the relations between $\mathbf{Q}_k$ and $\mathbf{R}_k$, on the one hand, and the corresponding values of $\mathbf{Q}$ and $\mathbf{R}$ for small sampling intervals Δt. Based on formula (5) we can notice that for small Δt values the discrete transition matrix $\mathbf{F}_k = \mathbf{I}$. After applying this conclusion to the matrix $\mathbf{Q}_k$ given by formula (8) we get (Brown & Hwang, 1997).

$$\mathbf{Q}_k = \iint\limits_{\Delta t \to 0} \mathbf{G}(\xi)E\left[\mathbf{u}(\xi)\mathbf{u}^T(\eta)\right]\mathbf{G}^T(\eta)d\xi d\eta \tag{30}$$

Then placing the equation (27) into the equation (30) and integrating for a small time interval Δt we get

$$\mathbf{Q}_k = \mathbf{G}\mathbf{Q}\mathbf{G}^T\Delta t \tag{31}$$

Deriving the equation relating $\mathbf{R}_k$ to $\mathbf{R}$ is not straightforward. In the continuous model the signal $\mathbf{v}(t)$ is the white noise and direct sampling returns the measuring noise with infinite variance. In order to get equivalent values at the discrete and continuous times it is assumed that the continuous measurements are averaged over the time interval Δt in the sampling process. The state variables $\mathbf{x}$ are not "white" and can be approximated as constant along this interval. Hence

$$\mathbf{y}_k = \frac{1}{\Delta t}\int_{t_{k-1}}^{t_k} y(t)dt = \frac{1}{\Delta t}\int_{t_{k-1}}^{t_k}\left[\mathbf{C}\mathbf{x}(t) + \mathbf{v}(t)\right]dt$$

$$\approx \mathbf{H}_k\mathbf{x}_k + \frac{1}{\Delta t}\int_{t_{k-1}}^{t_k}\mathbf{v}(t)dt \tag{32}$$

where $\mathbf{H}_k = \mathbf{C}(t_k)$. This way a new equivalent is obtained which defines the relation between the continuous time and discrete time domains

$$\mathbf{v}_k = \frac{1}{\Delta t}\int\limits_{\Delta t \to 0}\mathbf{v}(t)dt \tag{33}$$

From formula (10) we get

$$E\left[\mathbf{v}_k\mathbf{v}_k^T\right] = \mathbf{R}_k = \frac{1}{\Delta t^2}\iint\limits_{\Delta t \to 0} E\left[\mathbf{v}(u)\mathbf{v}^T(v)\right]du dv \tag{34}$$

Placing equation (28) into equation (34) and integrating we get the required relation

$$\mathbf{R}_k = \frac{\mathbf{R}}{\Delta t} \tag{35}$$

The amplification of the discrete Kalman filter is given by formula (23). After placing relation (35) into this formula we get

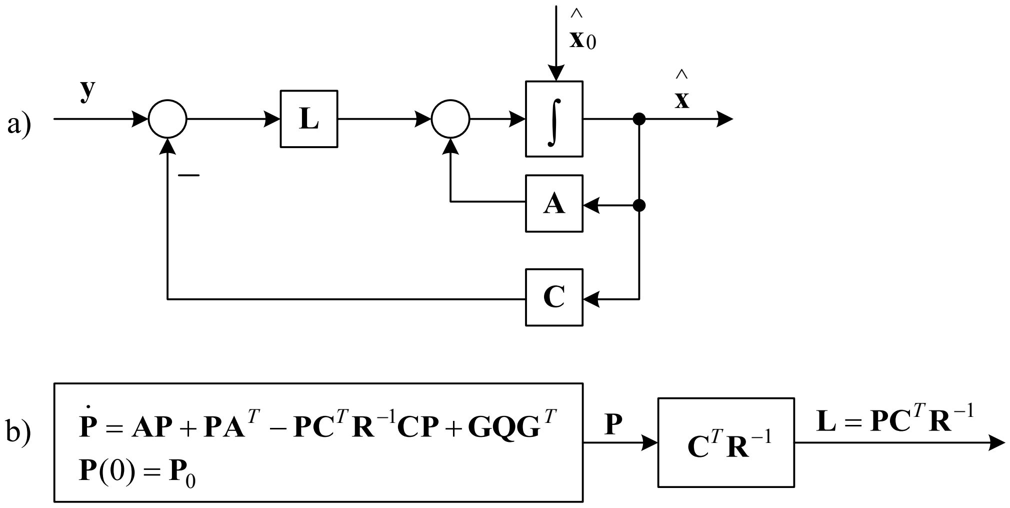

Figure 2. Block diagram of the continuous Kalman filter.

$$\mathbf{L}_k = \overline{\mathbf{P}}_k \mathbf{H}_k^T \left(\mathbf{H}_k \overline{\mathbf{P}}_k \mathbf{H}_k^T + \mathbf{R}/\Delta t\right)^{-1}$$

$$\approx \overline{\mathbf{P}}_k \mathbf{H}_k^T \mathbf{R}^{-1} \Delta t \tag{36}$$

as the second term inside the parentheses is dominating in formula (36)

$$\mathbf{R}/\Delta t >> \mathbf{H}_k \overline{\mathbf{P}}_k \mathbf{H}_k^T \tag{37}$$

When passing from the discrete form to the continuous form, the error covariance matrix between sampling times, given by formula (19), nears to $\overline{\mathbf{P}}_{k+1} \approx \mathbf{P}_k$ when $\Delta t \to 0$. Therefore only one error matrix $\mathbf{P}$ is in force in the continuous filter. Equation (36) takes the form

$$\mathbf{L}_k = \mathbf{P}\mathbf{C}^T \mathbf{R}^{-1} \Delta t \tag{38}$$

And, finally, when $\Delta t \to 0$, we arrive at the formula for determining the amplification $\mathbf{L}$ in the continuous Kalman filter (Brown & Hwang, 1997)

$$\mathbf{L} = \frac{\mathbf{L}_k}{\Delta t} = \mathbf{P}\mathbf{C}^T \mathbf{R}^{-1} \tag{39}$$

As the next step, the equation describing the estimation error covariance matrix is to be derived. Placing relation (25) into equation (19) we get

$$\overline{\mathbf{P}}_{k+1} = \mathbf{F}_k \left(\mathbf{I} - \mathbf{L}_k \mathbf{H}_k\right) \overline{\mathbf{P}}_k \mathbf{F}_k^T + \mathbf{Q}_k$$

$$= \mathbf{F}_k \overline{\mathbf{P}}_k \mathbf{F}_k^T - \mathbf{F}_k \mathbf{L}_k \mathbf{H}_k \overline{\mathbf{P}}_k \mathbf{F}_k^T + \mathbf{Q}_k \tag{40}$$

Then the discrete transition matrix $\mathbf{F}_k$ in equation (40) is substituted by its approximate given by formula (5) (Brown & Hwang, 1997)

$$\overline{\mathbf{P}}_{k+1} = \left(\mathbf{I} + \mathbf{A}\Delta t\right) \overline{\mathbf{P}}_k \left(\mathbf{I} + \mathbf{A}\Delta t\right)^T$$

$$- \left(\mathbf{I} + \mathbf{A}\Delta t\right) \mathbf{L}_k \mathbf{H}_k \overline{\mathbf{P}}_k \left(\mathbf{I} + \mathbf{A}\Delta t\right)^T + \mathbf{Q}_k$$

$$= \overline{\mathbf{P}}_k + \mathbf{A}\overline{\mathbf{P}}_k \Delta t + \overline{\mathbf{P}}_k \mathbf{A}^T \Delta t + \mathbf{A}\overline{\mathbf{P}}_k \mathbf{A}^T \Delta t^2$$

$$- \mathbf{L}_k \mathbf{H}_k \overline{\mathbf{P}}_k - \mathbf{A}\mathbf{L}_k \mathbf{H}_k \overline{\mathbf{P}}_k \Delta t - \mathbf{L}_k \mathbf{H}_k \overline{\mathbf{P}}_k \mathbf{A}^T \Delta t$$

$$- \mathbf{A}\mathbf{L}_k \mathbf{H}_k \overline{\mathbf{P}}_k \mathbf{A}^T \Delta t^2 + \mathbf{Q}_k \tag{41}$$

It can be seen from equation (38), than in equation (41) the matrix $\mathbf{L}_k$ is of an order of Δt. And, after removing all terms containing Δt of and order higher than one from equation (41) we get the simplified form (Brown & Hwang, 1997):

$$\overline{\mathbf{P}}_{k+1} = \overline{\mathbf{P}}_k + \mathbf{A}\overline{\mathbf{P}}_k \Delta t + \overline{\mathbf{P}}_k \mathbf{A}^T \Delta t - \mathbf{L}_k \mathbf{H}_k \overline{\mathbf{P}}_k + \mathbf{Q}_k \tag{42}$$

After substituting formula (38) for $\mathbf{L}_k$ and formula (31) for $\mathbf{Q}_k$ in equation (42) we arrive at

$$\overline{\mathbf{P}}_{k+1} = \overline{\mathbf{P}}_k + \mathbf{A}\overline{\mathbf{P}}_k \Delta t + \overline{\mathbf{P}}_k \mathbf{A}^T \Delta t$$

$$- \overline{\mathbf{P}}_k \mathbf{H}_k^T \mathbf{R}^{-1} \mathbf{H}_k \overline{\mathbf{P}}_k \Delta t + \mathbf{G}\mathbf{Q}\mathbf{G}^T \Delta t \tag{43}$$

Based on equation (43) we can get the following difference

$$\frac{\overline{\mathbf{P}}_{k+1} - \overline{\mathbf{P}}_k}{\Delta t} = \mathbf{A}\overline{\mathbf{P}}_k + \overline{\mathbf{P}}_k \mathbf{A}^T$$

$$- \overline{\mathbf{P}}_k \mathbf{H}_k^T \mathbf{R}^{-1} \mathbf{H}_k \overline{\mathbf{P}}_k + \mathbf{G}\mathbf{Q}\mathbf{G}^T \tag{44}$$

Then, after limiting $\Delta t \to 0$ and removing all subscripts and bars over matrices we get the matrix differential equation

$$\dot{\mathbf{P}} = \mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^T - \mathbf{P}\mathbf{C}^T \mathbf{R}^{-1} \mathbf{C}\mathbf{P} + \mathbf{G}\mathbf{Q}\mathbf{G}^T \tag{45}$$

with the initial condition

$$\mathbf{P}(0) = \mathbf{P}_0 \tag{46}$$

The last remaining step is to derive the state estimation equation given by formula (14). Placing the below given relation (47),

$$\overline{\mathbf{x}}_k = \mathbf{F}_{k-1} \hat{\mathbf{x}}_{k-1} \tag{47}$$

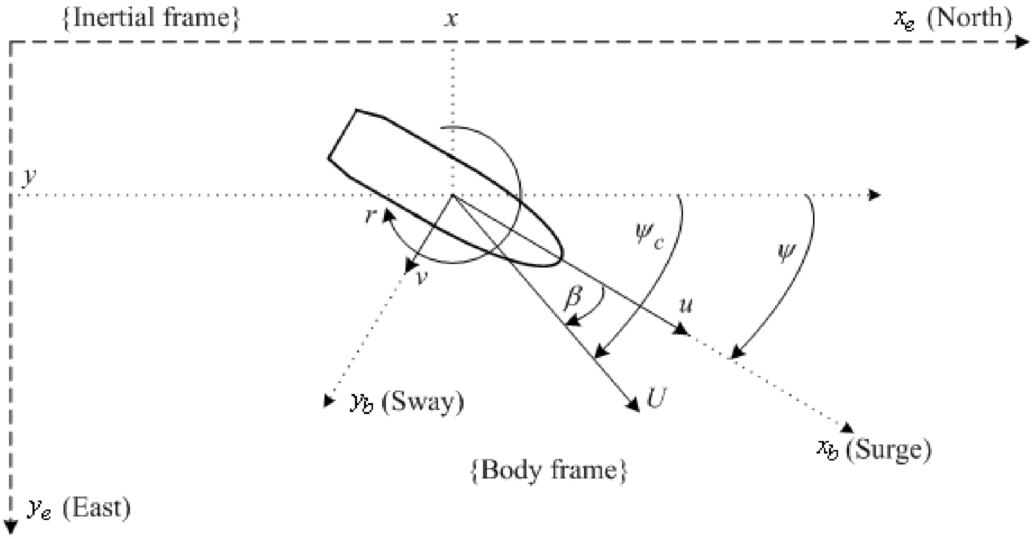

Figure 3. Variables used for describing ship motion.

derived from equation (16), into formula (14) makes it possible to arrive at the following form of equation (17)

$$\hat{\mathbf{x}}_k = \mathbf{F}_{k-1}\hat{\mathbf{x}}_{k-1} + \mathbf{L}_k\left(\mathbf{y}_k - \mathbf{H}_k\mathbf{F}_{k-1}\hat{\mathbf{x}}_{k-1}\right) \tag{48}$$

Here again, the matrix $\mathbf{F}_{k-1}$ is approximated by formula (4)

$$\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_{k-1} + \mathbf{A}\hat{\mathbf{x}}_{k-1}\Delta t + \mathbf{L}_k\left(\mathbf{y}_k - \mathbf{H}_k\hat{\mathbf{x}}_{k-1} - \mathbf{H}_k\mathbf{A}\hat{\mathbf{x}}_{k-1}\Delta t\right) \tag{49}$$

After removing all terms containing Δt of an order higher than one from equation (50) and observing that $\mathbf{L}_k = \mathbf{L}\Delta t$, the equation takes the form

$$\hat{\mathbf{x}}_k - \hat{\mathbf{x}}_{k-1} = \mathbf{A}\hat{\mathbf{x}}_{k-1}\Delta t + \mathbf{L}\Delta t\left(\mathbf{y}_k - \mathbf{H}_k\hat{\mathbf{x}}_{k-1}\right) \tag{50}$$

Finally, after dividing by Δt,

$$\frac{\hat{\mathbf{x}}_k - \hat{\mathbf{x}}_{k-1}}{\Delta t} = \mathbf{A}\hat{\mathbf{x}}_{k-1} + \mathbf{L}\left(\mathbf{y}_k - \mathbf{H}_k\hat{\mathbf{x}}_{k-1}\right) \tag{51}$$

reducing by $\Delta t \to 0$, and removing all subscripts and bars over variable matrices we get the matrix differential equation of continuous state estimation

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{L}\left(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}\right) \tag{52}$$

Equations (39), (45), (46) and (52) compose the continuous Kalman filter. They are shown in Fig. 1. Figure 1(a) shows a block diagram illustrating the principle of operation of the continuous Kalman filter. The input signal for the filter is the measured noised output signal of the object. Figure 1(b) presents the method of determining the optimal amplification **L**.

5 APPLYING THE KALMAN-BUCY FILTER FOR THE ESTIMATION OF SHIP MOTION VELOCITIES

The algorithm of the continuous Kalman-Bucy filter described in the previous section was applied for the estimation of ship motion parameters. The tests were performed on the training ship *Blue Lady* owned by the Foundation of Sailing Safety and Environment Protection in Ilawa. *Blue Lady* is the physical model, in scale 1:24, of a tanker designed for transporting crude oil. The overall length of *Blue Lady* is $L = 13.75$ m, the width is $B = 2.38$ m, and the mass is $m = 22.934\text{x}10^3$ [kg].

The ship sailing on the surface of the water region is considered a rigid body moving in three degrees of freedom. The ship position (x, y) and the ship course ψ in the horizontal plane with respect to the stationary, inertial coordinate system $\{x_e, y_e\}$ are represented by the vector $\boldsymbol{\eta}=[x,y,\psi]^T$. The second coordinate system $\{x_b, y_b\}$ is connected with the moving ship and fixed to its centre of gravity. Velocities of the moving ship are represented by the vector $\boldsymbol{\nu}=[u,v,r]^T$, where u is the longitudinal ship velocity (surge), v is the lateral velocity (sway), and r is the angular speed (yaw). These variables are shown in Fig. 3.

The position coordinates (x, y) are measured by DGPS (Differential Global Positioning System), while the ship course ψ is measured by the gyrocompass. These three measured state variables are collected in the vector $\boldsymbol{\eta}=[x,y,\psi]^T$. The three remaining state variables, composing the vector $\boldsymbol{\nu}=[u,v,r]^T$, are to be estimated.

The ship motion equations simply express the Newton's second law of motion in three degrees of freedom. These equations, formulated in the stationary coordinate system connected with the map of the water region, have the following form (Clarke, 2003).

$$m\ddot{x} = X \tag{53}$$

$$m\ddot{y} = Y \tag{54}$$

$$I_z\ddot{\psi} = N \tag{55}$$

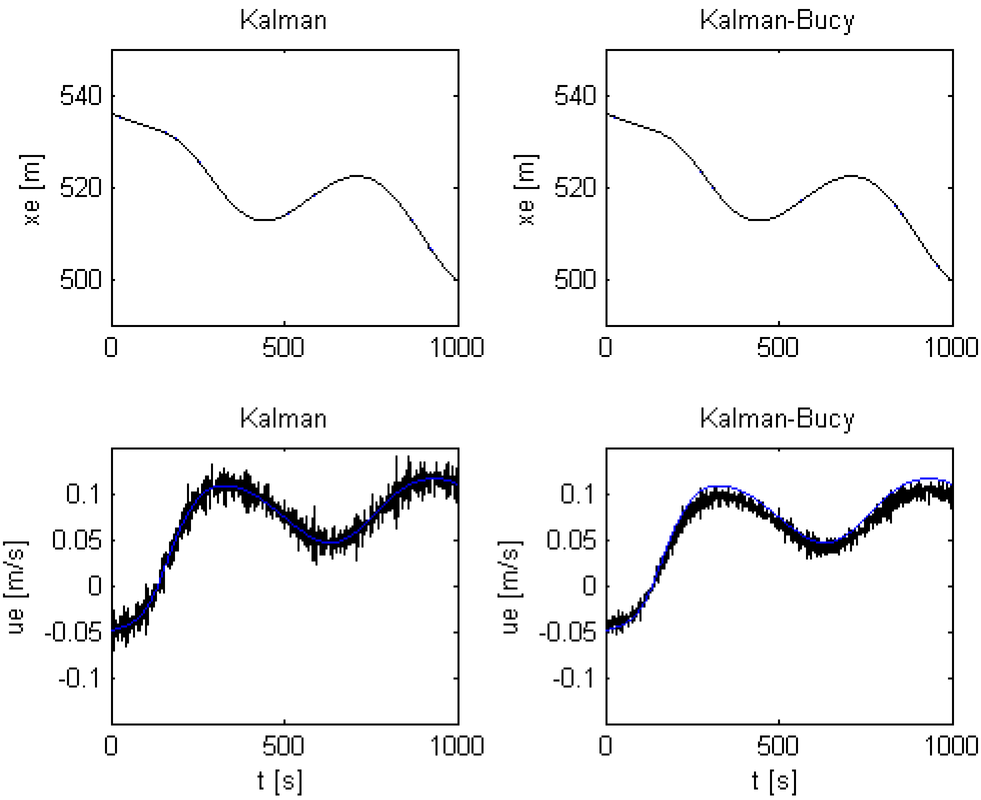

Figure 4. Simulation study: actual position with estimate and actual (black) and estimated (blue) velocity u in surge. Left-hand column – discrete Kalman filter, right-hand column – continuous Kalman-Bucy filter.

where X and Y are forces acting along the x_b and y_b axes, respectively, N is the torque, m is the mass of the ship, and I_z is the moment of inertia along the lateral axis directed downwards.

The above differential equations can be presented as three sets of dynamic equations having the following general form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{56}$$

$$y = \mathbf{C}\mathbf{x} \tag{57}$$

For each degree of freedom the matrices **A**, **B**, **C** are identical and take the form

$$\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix} \tag{58}$$

while the state vectors for consecutive states of freedom are the following

$$\mathbf{x}_1 = \begin{bmatrix} u_x \\ x \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} u_y \\ y \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} r \\ \psi \end{bmatrix} \tag{59}$$

where $u_x = dx/dt$, $v_y = dy/dt$, $r = d\psi/dt$ are velocities in the stationary coordinate system. The velocity vector $\mathbf{v}=[u,v,r]^T$ expressed in the moving coordinate system $\{x_b, y_b\}$, can be calculated based on the velocities determined in the stationary coordinate system $\{x_e, y_e\}$ and making use of the relation

$$\begin{bmatrix} u \\ v \\ r \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_x \\ v_y \\ r \end{bmatrix} \tag{60}$$

The continuous Kalman-Bucy filter implemented for estimating the motion parameters on the training ship Blue Lady worked based on the system and equations shown in Fig. 2. Moreover, for the purposes of the algorithm of the Kalman-Bucy filter, relevant values of the coefficients in the matrices **G**, **Q** and **R** were selected. For the first and second degree of freedom the following values were adopted:

$$\mathbf{G}_x = \mathbf{G}_y = \begin{bmatrix} 1 & 0 \\ 0 & 0.01 \end{bmatrix}, \ \mathbf{Q}_x = \mathbf{Q}_y = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix},$$

$$R_x = R_y = 0.01 \tag{61}$$

while for the third degree of freedom:

$$\mathbf{G}_\psi = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.01 \end{bmatrix}, \ \mathbf{Q}_\psi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ R_\psi = 0.1 \tag{62}$$

The covariances of the position coordinates measured by GPS were equal to $R_x = R_y = 0.01$while the covariances of the ship course measurement were equal to $R_\psi = 0.1$ and were determined based on the experimental tests done on the training ship *Blue Lady*.

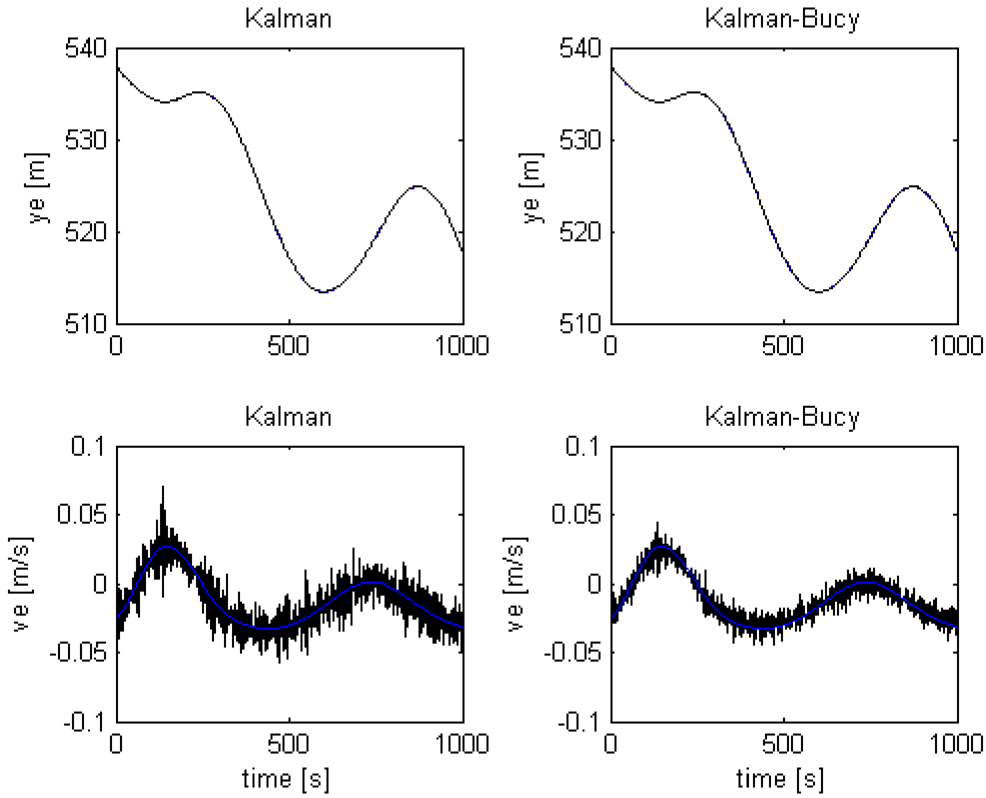

Figure 5. Simulation study: actual position with estimate and actual (black) and estimated (blue) velocity v in sway. Left-hand column – discrete Kalman filter, right-hand column – continuous Kalman-Bucy filter.

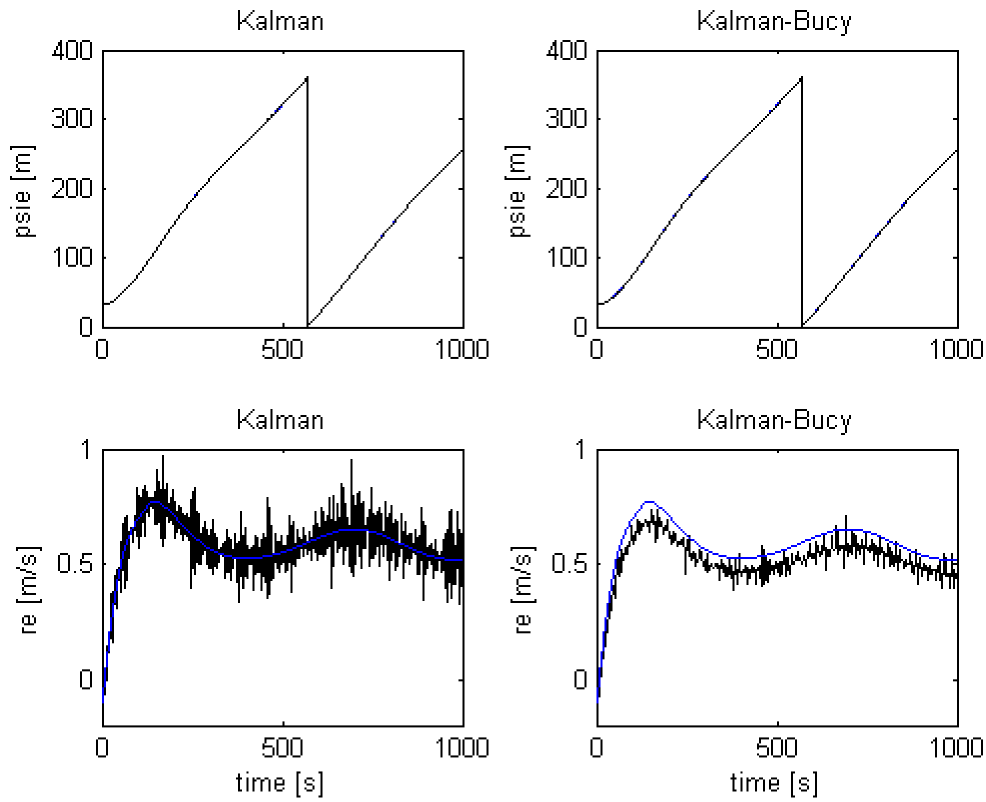

Figure 6. Simulation study: actual heading angle ψ with estimate and actual (black) and estimated (blue) angular rate r in yaw. Left-hand column – discrete Kalman filter, right-hand column – continuous Kalman-Bucy filter.

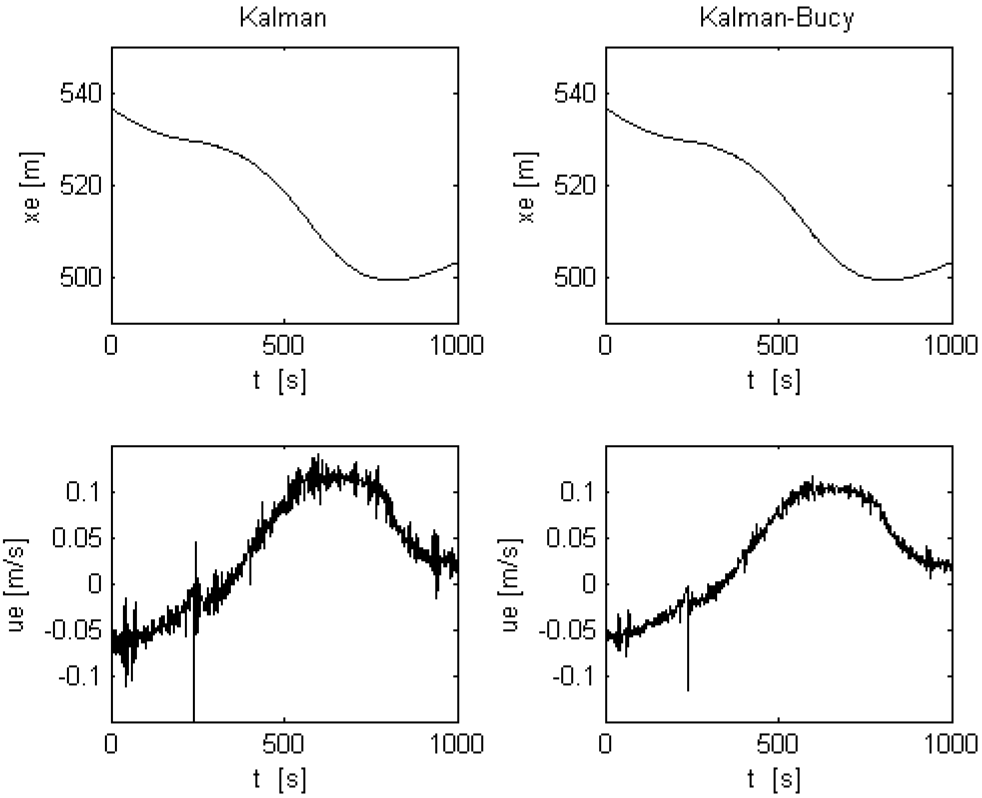

Figure 7. Experimental data: measured position with estimate and estimated velocity *u* in surge.
Left-hand column – discrete Kalman filter, right-hand column – continuous Kalman-Bucy filter.

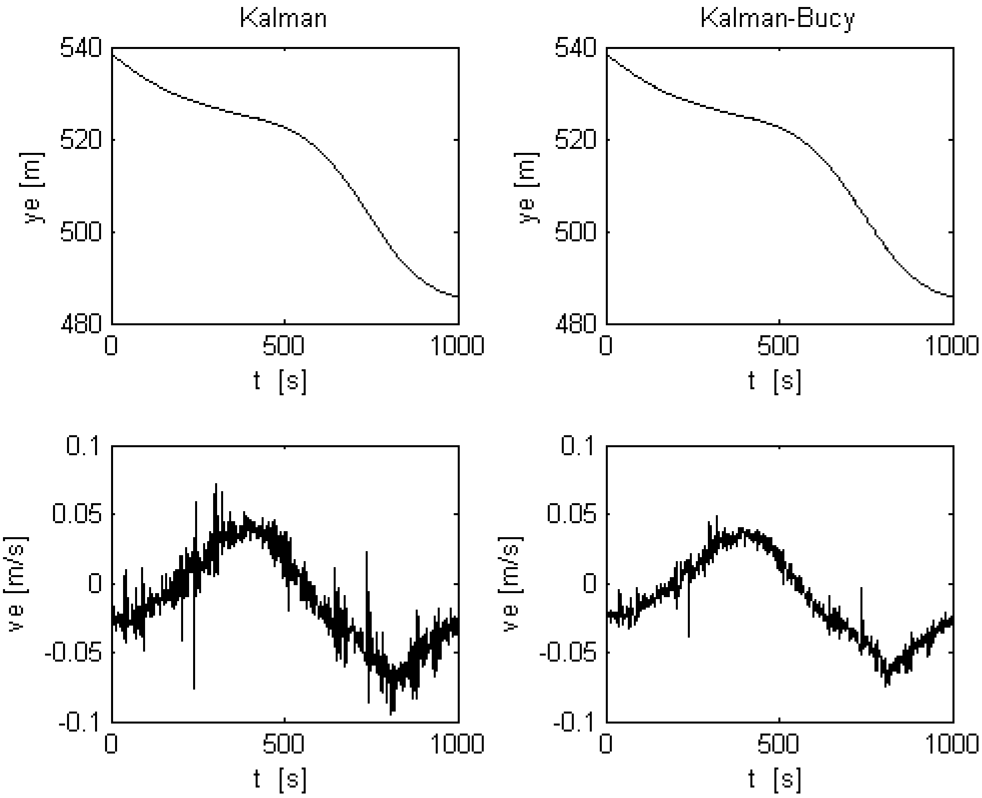

Figure 8. Experimental data: measured position with estimate and estimated velocity *v* in sway.
Left-hand column – discrete Kalman filter, right-hand column – continuous Kalman-Bucy filter.

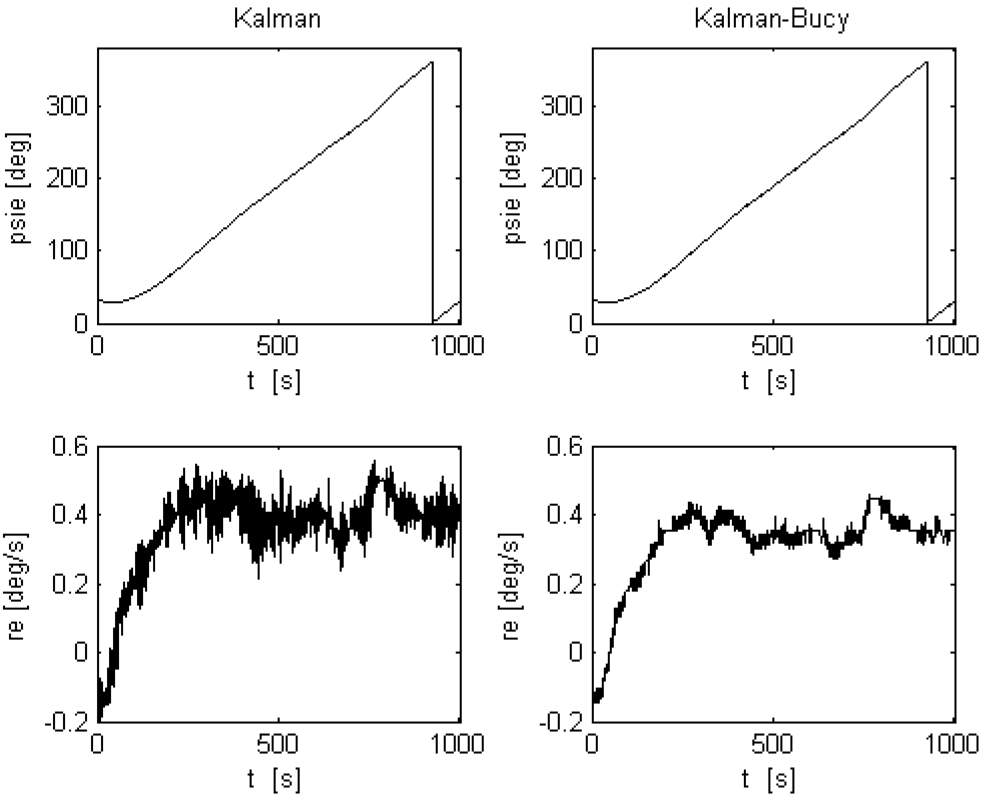

Figure 9. Experimental data: measured heading angle ψ with estimate and estimated angular rate r in yaw. Left-hand column – discrete Kalman filter, right-hand column – continuous Kalman-Bucy filter.

Initially, the simulation investigations were performed in the calculating environment Matlab/Simulink based on the mathematical model of the training ship *Blue Lady*, described in detail by (Gierusz, 2001; Gierusz. 2005). These investigations were performed at the presence of measurement noises, which were added to the positions and heading measurements in simulations and at the presence of the external disturbances. In simulation study assumed that on ship was acting wind with average speed equal 2 m/s and in direction 0 degrees. The simulation results are shown in Figs. 4-6. The actual and estimated velocities in surge, sway and yaw are shown in the bottom of plots.

After simulation tests, the algorithm of the discrete Kalman-Bucy filter was implemented on the training ship *Blue Lady* sailing on the lake Silm near Ilawa. The performed experimental tests aimed at testing the quality of filter operation and its resistance to disturbances. In order to provide good opportunities for comparison, the tests of the training ship Blue Lady were also performed using the discrete Kalman filter described by Tomera (Tomera, 2010).

The results recorded in the experimental tests are shown in Figs. 7 – 9. The diagrams in the left-hand column present the results obtained for the discrete Kalman filter while the time histories in the right-hand column refer to the investigations performed using the continuous Kalman-Bucy filter.

The presented diagrams reveal that the estimates of the position coordinates and the course are identical as the measured values. On the other hand, the correspondence between the time-histories of the estimated velocities is much worse, as their curves are not smooth and are burdened with relatively large errors. All inaccuracies in the measured values are reflected in the estimated velocity values.

6 REMARKS AND CONCLUSIONS

The article describes the method of deriving the algorithm of the continuous Kalman-Bucy filter based on the discrete Kalman filter. This algorithm does not take into account the model of ship dynamics, but only bases on the position coordinates x, y measured by GPS and the ship course angle ψ measured by the gyro-compass. The estimates of ship position coordinates x, y and ship course angle ψ shown in Figs. 7, 8 and 9 well correspond to the measured values. The correspondence is worse for velocity estimates determined for the moving coordinate system fixed to the ship and collected in the vector $\mathbf{v}=[u,v,r]^T$. The determined time-histories of these velocities are burdened with a noise of relatively high level, lower for the Kalman-Bucy controller than for the discrete Kalman filter. The shapes of the velocity estimate curves indicate that they need additional smoothing.

Additional difficulties in velocity estimation may appear when the instantaneous ship position coordinates measured by GPS reveal rapid changes. In this situation additional oscillations can be observed in the estimated velocities. Fortunately, in the sample results of investigations shown in Figs. 7 through 9 these difficulties were not recorded.

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