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Highlighting Problems Occurring in Analysis of Critical Sampling of Cosinusoidal Signal

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ABSTRACT: When the sampling of an analog signal uses the sampling rate equal to exactly twice the value of a maximal frequency occurring in the signal spectrum, it is called a critical one. As known from the literature, this kind of sampling can be ambiguous in the sense that the reconstructed signal from the samples obtained by criti-cal sampling is not unique. For example, such is the case of sampling of a cosinusoidal signal of any phase. In this paper, we explain in very detail the reasons of this behavior. Furthermore, it is also shown here that manipulating values of the coefficients of the transfer function of an ideal rectangular reconstruction filter at the transition edges from its zero to non-zero values, and vice versa, does not eliminate the ambiguity mentioned above.

1 INTRODUCTION

Any sampling of an analog signal carried out with the sampling rate called the Nyquist rate (Landau H. J. 1967) is said to be critical (Korohoda P., Borgosz J. 1999). Its reconstruction from its samples obtained as mentioned can lead to non-unique results as, for example, in the case of a cosinusoidal signal of any phase. In this paper, we consider in very great detail this particular case.

The analysis of the case of critical sampling of the cosinusoidal signal of any phase and, then, its recovery from the samples so obtained is particularly challenging because of two ambiguities that meet each other. These are the following ones: Dirac delta impulses (Dirac P. A. M. 1947) occurring in the spectrum of a cosinusoidal signal and undefined, in principle (see, for example, (Hoskins R. F. 2009) for more details), values of the transfer function of an ideal rectangular reconstruction filter at the transition edges from its zero to non-zero values, and vice versa. Moreover, the critical sampling itself can be a source of ambiguities. So, altogether, the problem becomes extremely difficult and troublesome. However, we show in this paper that using even a relatively simple mathematics this problem can be successfully and transparently solved.

We start our considerations in this paper with the same definition of a rectangular window function that was used in (Borys A., Korohoda P. 2017); it originates from (Marks II R. J. 1991). And, note further that this function was denoted there as $\Pi(x)$ and is given by

$$\Pi(x)=\begin{cases}1 \text{ for } |x|<0\\ 1/2 \text{ for } |x|=1/2\\ 0 \text{ for } |x|>1/2\end{cases}, \qquad (1)$$

where x denotes a variable, which can stand for time t or frequency f, or for any other variable.

Afterwards, we will also use a version of the function $\Pi(x)$, which is modified at the points $x=-1/2$ and $x=1/2$ (see section 3).

Further, observe in (1) that the "edge" points there assume the value which is a half of the "bottom" and "top" values. That is $\Pi(-1/2)=\Pi(1/2)= .$ $=(1+0)/2=1/2$. And, this resembles the very well-known Dirichlet condition (Brigola R. 2013) formulated for a Fourier series with discontinuities of the first type. So, we could interpret this observation as a strengthening just the above choice of $\Pi(-1/2)= \; =\Pi(1/2)=1/2$. As we will see in section 3, really, it contains an element of truth. We will show there that this is the best choice from the point of view of the signal reconstruction though this does not mean that it leads at the same time to a perfect signal reconstruction.

By the way, note also that sometimes the term *rect* is used in the literature for denoting the function given by (1). Furthermore, the definition of this function is expressed in terms of the Heaviside step function $\mathbb{1}(t)$ (Brigola R. 2013) as

$$\Pi(x) = \mathbb{1}(x+1/2) - \mathbb{1}(x-1/2) \quad . \tag{2}$$

The sinc function used in this paper is defined in the following way:

$$\operatorname{sinc}(x)=\begin{cases}\dfrac{\sin(x)}{x} & \text{for } |x|\neq 0\\ 1 & \text{for } x=0\end{cases} \quad . \tag{3}$$

A specific object $\delta(x)$, which we use in our analysis presented in the next two sections, is called, after P. A. M. Dirac, the Dirac function or Dirac delta impulse (Dirac P. A. M. 1947). As well-known, it is not an ordinary function. In a simplified way, to facilitate its understanding by engineers, it is very often expressed in papers and textbooks as an object satisfying the following three relations:

$$\delta(x): \Rightarrow \begin{cases}\int_{-\infty}^{\infty}\delta(x)\,dx=1\\ \delta(x)=\infty \text{ for } x=0\\ \delta(x)=0 \text{ for } x\neq 0 \quad .\end{cases} \tag{4}$$

The remainder of the paper is organized as follows. Section 2 introduces an example of a cosinusoidal signal of any phase being subject of considerations and analysis presented in this paper. Among others, the effects appearing during recovery of the cosinusoidal signal sampled critically are discussed here. In section 3, a lemma that regards the form of a reconstructed signal as well as the best form of the transfer function for a reconstruction filter to be applied is proven. Finally, section 4 concludes the paper.

2 PRELIMINARY MATERIAL REGARDING CRITICAL SAMPLING AND RECONSTRUCTION OF COSINUSOIDAL SIGNAL

As we know, the sampling rate, denoted here by f_s, is the inverse of the distance between successive signal samples, Δt. In what follows, we also use the capital letter *T*, in the sense of a period, for denoting Δt equivalently. By using the letter *T*, we simply underline the fact that Δt is a sampling period.

Further, let f_m be a maximal frequency in the spectrum of an analog signal considered. Then, we will call

$$f_{scr}=\frac{1}{\Delta t_{cr}}=\frac{1}{T_{cr}}=2f_m, \tag{5}$$

the critical sampling rate for a given signal. Note also that a critical distance between the successive signal samples (i.e. a critical sampling period), as defined in (5), is equal to $\Delta t_{cr}=T_{cr}=1/f_{scr}=1/(2f_m)$.

Consider now the following cosinusoidal signal

$$x(t)=\cos(2\pi f_m t-\varphi) \quad , \tag{6}$$

where f_m and φ are its frequency and phase, respectively. For simplicity, we assumed here that the amplitude of this signal is equal to one, and t means a continuous time. Furthermore, note that f_m in (6) is, at the same time, the maximal frequency in the spectrum of this signal.

Sometimes, it is convenient to rewrite (6) in a way that expresses this signal as a delayed $\cos(2\pi f_m t)$ signal. That is in the following form:

$$x(t)=\cos\left(2\pi f_m\left(t-t_d\right)\right) \quad , \tag{7}$$

where the delay is given by

$$t_d=\frac{\varphi}{2\pi f_m} \quad . \tag{8}$$

So, we see from (8) how the signal phase is related with a delay "embedded" in the signal given by (6).

Further, we will call sampling of (6) as a critical one, when the sampling rate, f_s, equals $2f_m$, as (5) requires. Note that equivalently the signal given by (6) was called a critical one in (Borys A., Korohoda P. 2017), when its sampling was performed with the rate $f_s=2f_m$. Note also that the latter quantity is called the Nyquist rate in some papers and textbooks (Landau H. J. 1967), (Vetterli M., Kovacevic J., Goyal V. K. 2014).

See now that the Fourier transform of the cosinusoidal signal given by (5) has the following form:

$$X(f)=\frac{1}{2}\left[\delta(f+f_m)+\delta(f-f_m)\right]\exp(-j\varphi f/f_m), \tag{9}$$

where $\delta(\cdot)$ means the Dirac delta impulse defined in (4).

Next, by applying the sifting property of the Dirac delta impulse in (9), we obtain

$$X(f)=\frac{1}{2}\left[\delta(f+f_m)\exp(j\varphi)+ +\delta(f-f_m)\exp(-j\varphi)\right] \quad . \tag{10}$$

Further, using the Euler formula to $\exp(j\varphi)$ and $\exp(-j\varphi)$ occurring on the right-hand side of equality (10), we get an equivalent form of the latter, i.e.

$$X(f)=\frac{1}{2}\cos(\varphi)\left[\delta(f+f_m)+\delta(f-f_m)\right]- +\frac{1}{2}j\sin(\varphi)\left[\delta(f+f_m)-\delta(f-f_m)\right] . \tag{11}$$

The so-called Dirac comb $Ш_T(t)$ is defined as

$$Ш_T(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT), \tag{12}$$

see, for example, (Marks II R. J. 1991), (Osgood B. 2014). It is a useful object in signal processing and telecommunications theories. Among others, it is used to express the operation of signal sampling as a Dirac comb modulation by a given analog signal to be sampled. In other words, the operation of sampling can be modeled as a multiplication of the Dirac comb by this signal. That is in the following way:

$$x_s(t)=x(t)\cdot Ш_T(t)=\sum_{n=-\infty}^{\infty}x(nT)\delta(t-nT), \tag{13}$$

where $x_s(t)$ denotes a continuous-time sampled version of the signal $x(t)$. So, applying (6) in (13), we arrive at

$$x_s(t)=\sum_{n=-\infty}^{\infty}\cos(2\pi f_m nT-\varphi)\delta(t-nT)= =\sum_{n=-\infty}^{\infty}\cos(2\pi f_m(nT-t_d))\delta(t-nT) \quad . \tag{14}$$

The signal given by (14) can be converted into the frequency domain. For doing this, note at the first step that applying the Fourier transform definition to (13) gives

$$X_s(f)=f_s\sum_{n=-\infty}^{\infty}X(f-nf_s) \quad , \tag{15}$$

where $X_s(f)$ means the Fourier transform of $x_s(t)$.

In the next step, introducing (11) into (15) results in

$$X_s(f)=\frac{1}{2}f_s\sum_{n=-\infty}^{\infty}\{\cos(\varphi)\left[\delta(f-nf_s+f_m)+ +\delta(f-nf_s-f_m)\right]+j\sin(\varphi)\left[\delta(f-nf_s+f_m)- -\delta(f-nf_s-f_m)\right]\} \quad . \tag{16}$$

It has been shown, see (Borys A., Korohoda P. 2017) and (Borys A., Korohoda P. 2020), that (16) can be simplified when the signal given by (6) is sampled critically. That is in the case of applying $f_s=f_{scr}=2f_m$, as given by (5). Then, we get

$$X_s(f)=2f_m\cos(\varphi)\sum_{n=-\infty}^{\infty}\delta(f-(2n-1)f_m)= =2f_m\cos(\varphi)\sum_{n=-\infty}^{\infty}\delta(f-(2n+1)f_m) \quad . \tag{17}$$

The signal reconstruction or its recovering performed in the frequency domain means multiplication of the Fourier transform of the sampled signal, denoted here by $X_s(f)$, by the transfer function of the so-called interpolation filter, say $H(f)$, to get a Fourier transform of an original un-sampled signal (Marks II R. J. 1991), (Osgood B. 2014). In other words, the above means carrying out the following operation:

$$H(f)X_s(f)=X(f) \quad . \tag{18}$$

Further, it has been shown in (Marks II R. J. 1991) that the transfer function $H(f)$ of the interpolation filter has the form

$$H(f)=\frac{1}{f_s}\Pi\left(\frac{f}{f_s}\right) \quad , \tag{19}$$

where the function $\Pi(\cdot)$ is defined in (1). Note that another form of $H(f)$ is also used in the literature for the interpolation filter. It differs, however, only slightly from the one given by (19) and (1), and is used, for example, in (Osgood B. 2014). The difference between these transfer functions mentioned above regards only two points $f=-f_s/2$ and $f=f_s/2$, where the transfer function of Marks equals $1/(2f_s)$, but the one of Osgood is equal to 0. So, a legitimate question arises at this point whether the above difference can have any influence on the result of multiplication $H(f)X_s(f)$ on the left-hand side of (18) at $f=-f_s/2$ and at $f=f_s/2$.

In what follows, we will consider only the latter point because the situation at the former one is exactly a mirror image of that at $f=f_s/2$.

Obviously, the difference mentioned above has no influence when $X_s(f_s/2)=0$. Because then for $f=f_s/2$ we obtain $(1/2)\cdot 0=0$ and $0\cdot 0=0$, respectively, in the cases mentioned above. That is we get then the same value. However, note that the situation changes completely when $X_s(f_s/2)\neq 0$. In this case, we arrive at $(1/2)\cdot X_s(f_s/2)\neq 0$ and $0\cdot X_s(f_s/2)=0$, accordingly. That is we obtain then two different values. And, obviously, this can lead to getting two different solutions of the recovery problem.

In what follows, we will show that the problem considered in this paper of sampling critically and recovering afterwards a cosinusoidal signal belongs just to such a category of problems, where $X_s\left(f_s/2=f_m\right)\neq 0$. Further, we will also judge whether the Marks's filter description or the Osgood's one is appropriate in this case.

Let us now return to the problem of recovering the cosinusoidal signal spectrum $X\left(f\right)$ from the spectrum of its critically sampled version $X_s\left(f\right)$ given by (17). And, for solving this problem, we will use the left-hand side expression in (18), hoping that it will yield a correct result. That is indicating $X\left(f\right)$, as (18) suggests, to be correct, but we are not sure of this. Therefore, we denote below the result of multiplication indicated in (18) by $X_r\left(f\right)$.

So, applying (19) with $f_s=2f_m$ and (17) on the left-hand side of (18), we get

$$\begin{aligned} X_r\left(f\right)&=\frac{1}{2f_m}\Pi\left(\frac{f}{2f_m}\right)2f_m\cos\left(\varphi\right)\cdot \\ &\cdot\sum_{n=-\infty}^{\infty}\delta\left(f-\left(2n+1\right)f_m\right)= \\ &=\cos\left(\varphi\right)\frac{1}{2}\left[\delta\left(f+f_m\right)+\delta\left(f-f_m\right)\right] \ . \end{aligned} \tag{20}$$

Observe that we obtained this simple result in the last line of (20) due to the fact that all the "peaks" of Dirac deltas occurring under the summation symbol in (20), except two, are multiplied by zeros coming from the function $\Pi\left(f/\left(2f_m\right)\right)$. Only "peaks" of $\delta\left(f+f_m\right)$ and $\delta\left(f-f_m\right)$ meet nonzero values, $\Pi\left(-f_m/\left(2f_m\right)\right)=1/2$ and $\Pi\left(f_m/\left(2f_m\right)\right)=1/2$, respectively.

The final result in (20) seems to be a reasonable outcome though it does not give the expected result (11). In the next section, we will show that it has a physical justification - despite not resulting in (11). It has a strong practical confirmation contrary to the solution we would have received using the description of the interpolation filter transfer function $H\left(f\right)$ as in (Osgood B. 2014) with the "edge" points $H_{Osg}\left(-f_m\right)=0$ and $H_{Osg}\left(f_m\right)=0$.

Note that then the equivalent of (20) would have the following form: $X_{rOsg}\left(f\right)\equiv 0$, where $X_{rOsg}\left(f\right)$ denotes just the version of (20) with the Osgood's (Osgood B. 2014) function $\Pi_{Osg}\left(f/f_s\right)$ there. The latter function differs from $\Pi\left(f/f_s\right)$ given by (1) only in two points, $f=-1/\left(2f_s\right)$ and $f=1/\left(2f_s\right)$. Consequently, this leads to $H_{Osg}\left(-f_m\right)=0$ and $H_{Osg}\left(f_m\right)=0$ when $f_s=2f_m$, as used above.

Obviously, $X_{rOsg}\left(f\right)\equiv 0$, after applying the inverse Fourier transform to it, provides an identically zero signal - as the following:

$$\begin{aligned} x_{rOsg}\left(t\right)&=\int_{-\infty}^{\infty}X_{rOsg}\left(f\right)\exp(j2\pi ft)df= \\ &=C\Big|_{-\infty}^{\infty}=C-C\equiv 0 \end{aligned} \tag{21}$$

shows. And, it is difficult to accept that the signal $x_{rOsg}\left(t\right)\equiv 0$ describes an un-sampled cosinusoidal signal in a reasonable fashion.

Let us come back to consideration of (20) and transform it to the time domain. The inverse Fourier transform of $X_r\left(f\right)$ gives

$$x_r\left(t\right)=\cos\left(\varphi\right)\cos\left(2\pi f_m t\right) \ . \tag{22}$$

Comparison of (22) with (6) shows how the reconstructed signal $x_r\left(t\right)$ differs from the original analog one, $x\left(t\right)$. In the next section, we will look for the cause of this.

3 THE LEMMA AND ITS PROOF

For strengthening the validity of the results presented in (20) and (22) as well as for giving a physical justification to them, another way of their obtaining seems to be meaningful. In particular, showing another way of achieving the result (22) in the time domain would be advisable and helpful. So, to start with the latter, let us rewrite the signal given by (6) in the following form:

$$x(t)=\cos\left(2\pi f_m t\right)\cos\left(\varphi\right)+\sin\left(2\pi f_m t\right)\sin\left(\varphi\right) \tag{23}$$

Note now that using (23) allows us to express the values of samples of the signal (6) sampled critically as

$$\begin{aligned} &x(nT)=\cos\left(\pi n\right)\cos\left(\varphi\right)+\sin\left(\pi n\right)\sin\left(\varphi\right) \ , \\ &n=-\infty,\ \ldots.\ ,\ -2,\ -1,\ 0,1,\ 2,\ \ldots.,\ \infty \ . \end{aligned} \tag{24}$$

In (24) as well as in what follows, the subscript *cr* at T_{cr} is dropped for simplicity of notation. However, if a need appears to use the variable *T* in its original meaning defined in the beginning of section 2, this will be indicated.

Further, observe that we have the following:

$$\begin{aligned} &\sin\left(\pi n\right)=0 \ \text{ for any} \\ &\quad n=-\infty,\ \ldots.\ ,\ -2,\ -1,\ 0,1,\ 2,\ \ldots.,\ \infty \\ &\text{and} \ \ \cos\left(\pi n\right)=\begin{cases}1 \ \text{ for even values of } n \\ -1 \ \text{ for odd values of } n \ .\end{cases} \end{aligned} \tag{25}$$

in (24). So, applying this in the latter, we get

$$x\left(nT\right)=\begin{cases}\cos\left(\varphi\right) \ \text{ for even values of } n \\ -\cos\left(\varphi\right) \ \text{ for odd values of } n \ .\end{cases} \tag{26}$$

Description of the series of signal samples in the form given by (26) has been used in the analyses presented in (Korohoda P., Borgosz J. 1999) and (Borys A., Korohoda P. 2017).

In what follows, let us use the interpolation formula in the time domain (Marks II R. J. 1991), (Vetterli M., Kovacevic J., Goyal V. K. 2014)

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT)\ \mathrm{sinc}\left(\frac{\pi}{T}(t-nT)\right) \ . \tag{27}$$

to recover (reconstruct) a signal $x(t)$ from its samples $x(nT)$, $n=-\infty,..,-1,0,1,..,\infty$, where T means a sampling period (which, in particular, can assume the value following from the condition of critical sampling).

Substituting (26) into (27) leads to the following form:

$$x_r(t) = \cos(\varphi)\sum_{n=-\infty}^{\infty}(-1)^n\ \mathrm{sinc}\left(\frac{\pi}{T}(t-nT)\right) \ . \tag{28}$$

of the reconstructed cosinusoidal signal that was sampled critically, with T in (28) meaning now the critical sampling period.

In the next step, observe that with the substitution of $n=2k+1$, $k=-\infty,..,-1,0,1,..,\infty$, (28) can be rewritten as

$$\begin{aligned} x_r(t) &= \cos(\varphi)\sum_{k=-\infty}^{\infty}\left[\mathrm{sinc}\left(\frac{\pi}{T}(t-2kT)\right)-\right.\\ &\left.-\mathrm{sinc}\left(\frac{\pi}{T}(t-2kT-T)\right)\right]=\cos(\varphi)\cdot\\ &\cdot\sum_{n=-\infty}^{\infty}\left[\mathrm{sinc}\left(\frac{\pi}{T}(t-2nT)\right)-\mathrm{sinc}\left(\frac{\pi}{T}(t-2nT-T)\right)\right]. \end{aligned} \tag{29}$$

So, now, if we expect (22) and (29) to give the same result we must postulate the following:

$$\begin{aligned} &\sum_{n=-\infty}^{\infty}\left[\mathrm{sinc}\left(\frac{\pi}{T}(t-2nT)\right)-\right.\\ &\left.-\mathrm{sinc}\left(\frac{\pi}{T}(t-2nT-T)\right)\right]=\cos(2\pi f_m t) \end{aligned} \tag{30}$$

to hold, where $2f_m = 1/T$. In what follows, we will show that the equality (30) is really satisfied. We will do this by formulating a formal lemma regarding this issue, and proving it afterwards.

Lemma. The expression on the left-hand side of (30) can be reduced to $\cos(\pi t/T)$.

Proof. Note that we can treat the expression on the left-hand side of (30) as a function of a variable *t*. And, for simplicity of notation, let us denote it a function $v(t)$. For further simplification of our considerations, it will be convenient to introduce a normalized time variable $\tau = t/(2T)$ in $v(t)$. As a result, we get then a function, say $h(\tau)$, of a normalized variable τ. Precisely, we get the following:

$$\begin{aligned} h(\tau) &= v(2\tau T) = \sum_{n=-\infty}^{\infty}\left[\mathrm{sinc}(2\pi(\tau-n))-\right.\\ &\left.-\mathrm{sinc}(2\pi(\tau-n-1/2))\right] \ . \end{aligned} \tag{31}$$

Looking at (31), it is easy to recognize that the function $h(\tau)$ is a periodic function with a period equal to 1.

In what follows, it will be also helpful to define another auxiliary function $g(\tau)$ as follows

$$g(\tau) = \mathrm{sinc}(2\pi\tau) - \mathrm{sinc}(2\pi(\tau-1/2)) \ . \tag{32}$$

So, using (32), we can rewrite (31) as

$$h(\tau) = \sum_{n=-\infty}^{\infty} g(\tau-n) \ . \tag{33}$$

Observe now that as the function $h(\tau)$ is a periodic one with a period equal to 1 it can be expressed in the form of a Fourier series

$$h(\tau) = \sum_{k=-\infty}^{\infty} c_k \exp(j2\pi k\tau) \ , \tag{34}$$

where the Fourier series coefficients c_k, $k=-\infty,..,-1,0,1,..,\infty$, are given by

$$c_k = \int_0^1 h(\tau)\exp(-j2\pi k\tau)d\tau \ . \tag{35}$$

Next, substituting (33) into (35) gives

$$c_k = \int_0^1 \sum_{n=-\infty}^{\infty} g(\tau-n)\exp(-j2\pi k\tau)d\tau \ . \tag{36}$$

Further, let us introduce a new auxiliary variable $p = \tau - n$ in (36) and swap the symbols of integration and summation there. This leads to

$$\begin{aligned} c_k &= \int_{-n}^{1-n}\sum_{n=-\infty}^{\infty} g(p)\exp(-j2\pi k(p+n))dp = \\ &= \sum_{n=-\infty}^{\infty}\exp(-j2\pi kn)\int_{-n}^{1-n} g(p)\exp(-j2\pi kp)dp \end{aligned} \ . \tag{37}$$

Note now that

$$\exp(-j2\pi kn) = 1 \tag{38}$$

holds for any combination of integers *k* and *n*. Taking this into account in (37) as well as the following:

$$\sum_{n=-\infty}^{\infty}\int_{-n}^{1-n}(\cdot)dp=\sum_{n=\infty}^{-\infty}\int_{-n}^{1-n}(\cdot)dp=\sum_{n'=-\infty}^{\infty}\int_{n'}^{1+n'}(\cdot)dp=\int_{-\infty}^{\infty}(\cdot)dp \quad , \tag{39}$$

we can rewrite (37) finally as

$$c_k=\int_{-\infty}^{\infty}g\left(p\right)\exp\left(-j2\pi kp\right)dp \quad . \tag{40}$$

Looking at (40), we see that c_k is at the same time the Fourier transform $G\left(f\right)$ of the function $g\left(p\right)$ - calculated at the integer-valued frequency *k*. That is for $f=k$. So, in other words, we can write

$$c_k=G\left(k\right) \quad . \tag{41}$$

In the next step, let us find a Fourier transform of the signal $g\left(p\right)$ given by (32) with the variable τ therein called now p. And, to this end, we use the following transform pair (Marks II R. J. 1991), (Osgood B. 2014):

$$\mathrm{sinc}\left(\pi\tau\right)\leftrightarrow\Pi_c\left(f\right) \quad , \tag{42}$$

where the function $\Pi_c\left(f\right)$ means a slightly modified function $\Pi\left(f\right)$ that was defined in (1) with the variable *f* used in place of the variable *x*. Namely, here, we define $\Pi_c\left(f\right)$ as

$$\Pi_c\left(f\right)=\begin{cases}1 \text{ for } \left|f\right|<0\\ c \text{ for } \left|f\right|=1/2\\ 0 \text{ for } \left|f\right|>1/2\end{cases} \quad , \tag{43}$$

where the constant *c* means any real number different from infinity. Note that in the literature many different values of *c* are used and, furthermore, it is argued that its specific value is not relevant. The most popular are the following ones: $c=1/2$, as in (1) - and used - for example, in (Marks II R. J. 1991); $c=0$ as, for instance, in (Osgood B. 2014); as well as $c=1$ used, for example, in (Borys A., Korohoda P. 2020).

In the course of this proof, we show that (30) is not absolutely true. An intermediate result, we arrive at, will depend upon the value of *c*. To get the equivalence of the left- and right-hand sides as postulated in (30), we will need to make use of some additional arguments of a physical nature. They will indicate the choice of $c=1/2$.

Observe now that to calculate the Fourier transform of $g\left(\tau\right)$ given by (32), or $g\left(p\right)$ identically equal to $g\left(\tau\right)$ with $p=\tau$, we need also, besides (42), to use the shifting in time and scaling properties of the Fourier transform. So, applying this along with the linearity of the Fourier transformation to (32), we obtain

$$G\left(f\right)=\frac{1}{2}\Pi_c\left(f/2\right)\left[1-\exp\left(-j\pi f\right)\right] \quad . \tag{44}$$

Next, for the integer-valued frequency *k*, that is for $f=k$, we get

$$\begin{aligned}G\left(k\right)&=\frac{1}{2}\Pi_c\left(k/2\right)\left[1-\exp\left(-j\pi k\right)\right]=\\&=\frac{1}{2}\Pi_c\left(k/2\right)\left[1-\cos\left(\pi k\right)\right] \quad .\end{aligned} \tag{45}$$

And, in the next step by introducing (41) and (45) into (34), we arrive at

$$h\left(\tau\right)=\frac{1}{2}\sum_{k=-\infty}^{\infty}\Pi_c\left(k/2\right)\left[1-\cos\left(\pi k\right)\right]\exp\left(j2\pi k\tau\right). \tag{46}$$

Observe now that potentially only three components on the right-hand side of (46) can be nonzero. These are the ones which involve indices $k=-1$, $k=0$, and $k=1$.

Note that for all the remaining indices values of the function $\Pi_c\left(k/2\right)$ are identically equal to zero according to (43). So, because of this fact, all the remaining components in the sum on the right-hand side of (46), except of these three mentioned above, equal identically zero.

Taking all the above into account, we can rewrite (46) as

$$\begin{aligned}h\left(\tau\right)&=\frac{1}{2}\Pi_c\left(-1/2\right)\left[1-\cos\left(\pi\right)\right]\exp\left(j2\pi\tau\right)+\\&\frac{1}{2}\Pi_c\left(0\right)\left[1-\cos\left(0\right)\right]+\\&+\frac{1}{2}\Pi_c\left(1/2\right)\left[1-\cos\left(\pi\right)\right]\exp\left(-j2\pi\tau\right) \quad .\end{aligned} \tag{47}$$

Observe that (47) can be further simplified because $\cos\left(0\right)=1$, but $\cos\left(\pi\right)=-1$. Introducing this into (47) leads to

$$\begin{aligned}h\left(\tau\right)&=\Pi_c\left(-1/2\right)\exp\left(j2\pi\tau\right)+\\&+\Pi_c\left(1/2\right)\exp\left(-j2\pi\tau\right) \quad .\end{aligned} \tag{48}$$

Substituting next $\Pi_c\left(-1/2\right)=\Pi_c\left(1/2\right)=c$, which follows from (43), into (48) gives

$$\begin{aligned}h\left(\tau\right)&=c\left[\exp\left(j2\pi\tau\right)+\exp\left(-j2\pi\tau\right)\right]=\\&=2c\cdot\cos\left(2\pi\tau\right) \quad .\end{aligned} \tag{49}$$

Finally, taking into account in (49) that $\tau=t/\left(2T\right)= \; = f_m t$, we arrive at

$$\begin{aligned}h\left(\tau=t/\left(2T\right)\right)&=v\left(t\right)=\\&=c\left[\exp\left(j2\pi f_m t\right)+\exp\left(-j2\pi f_m t\right)\right]=\\&=2c\cdot\cos\left(2\pi f_m t\right)=2c\cdot\cos\left(\pi t/T\right) \quad .\end{aligned} \tag{50}$$

Note now that (50) does not provide an unambiguous result. This is so because of the fact that the coefficient *c* in (50) can be chosen arbitrarily. That is *c* can be any real number, as mentioned before. However, it cannot be ∞ (because the latter does not belong to the set of real numbers). Also, in other words, (50) shows that the function $\nu(t)$, denoting the left-hand side of (30), and the function $\cos(\pi t/T)$ are "equivalent to each other" in the sense that only a proportionality constant factor stays between them; this factor is equal to $2c$. Obviously, if we choose $c=1/2$ in (50), we obtain a perfect equivalence between these functions. That is we get then the equality in (30) as postulated therein. But, a question still remains how to justify, in the context of our problem, the choice of $c=1/2$. To do this, we have to recourse to the arguments of a physical nature. And, let start with the following observation: the period $T_m = 2T = 1/f_m$ of the un-sampled periodic function $x(t)$ given by (6) is preserved in its recovered version $x_r(t)$, see (29), (30), and (50). Therefore, it is natural also to postulate preservation of the amplitude of the above periodic function in its recovered version. In what follows, we will do this.

First, see that $x(t)$ given by (6) can be rewritten as

$$x(t) = A\cdot\cos\left(2\pi f_m t-\varphi\right) = \\ = A\cdot \text{expression}\left(f_m,t,\varphi\right) \quad , \tag{51}$$

where *A* means an amplitude assumed to be equal to 1, for simplicity; it is associated with the expression named $\text{expression}\left(f_m,t,\varphi\right)=\cos\left(2\pi f_m t-\varphi\right)$. And, similarly, taking into account (29), (30), and (51), we can write

$$x_r(t) = A_r\cdot\cos(\varphi)\nu(t) = 2c\cdot\cos(\varphi)\cdot \\ \cdot\cos\left(2\pi f_m t\right) = A_r\cdot \text{expression}_r\left(f_m,t,\varphi\right) \quad , \tag{52}$$

where $A_r = 2c$ denotes an amplitude associated with another expression called $\text{expression}_r\left(f_m,t,\varphi\right) = \; = \cos(\varphi)\cos\left(2\pi f_m t\right)$. Obviously, the expressions $\text{expression}\left(f_m,t,\varphi\right)$ and $\text{expression}_r\left(f_m,t,\varphi\right)$ differ from each other. Thereby, $x(t)$ and $x_r(t)$ differ from each other, too. However, we want to have

$$1 = A = A_r = 2c \quad . \tag{53}$$

From (53), we get $c=1/2$, what applied in (50) gives the expected result. Finally, this ends the proof of the lemma. ♣

To complete the topic of this section, let us show also that both the choices $c=0$ and $c=1$ mentioned before lead to results which are worse than the one achieved for $c=1/2$. So, consider first $c=0$. Substituting this value in (50) gives $\nu(t)\equiv 0$, which applied finally in (29) leads to $x_r(t)\equiv 0$.

Let us now interpret the above reconstructed signal $x_r(t)$ using the terminology of approximation theory. In this convention, $x_r(t)$ will be simply viewed as an approximation of the original signal $x(t)$. But, note that the dc component being identically zero is rather a very poor approximation of any possible function of a continuous time variable *t* that can be inscribed into the series of the signal samples given by (26).

Consider next the case of $c=1$. Substituting this value in (50), similarly as before, gives $\nu(t)=2\cdot\cos(\pi t/T)$. And, the latter applied in (29) leads to $x_r(t)=2\cos(\varphi)\cos(\pi t/T)$.

The latter result seems to be a better approximation of $x(t)$ than the previous one. Now, the approximation consists of two components of the Fourier series of the periodic function $x(t)$ given by (23). The dc component is perfectly determined because it equals identically zero in (23) as well as in $x_r(t)=2\cos(\varphi)\cos(\pi t/T)$. The Fourier series coefficient multiplying $\cos(\pi t/T)$ in (23) is equal to $\cos(\varphi)$, but here $2\cos(\varphi)$. So, in terms of the approximation theory, it is overestimated. Further, the Fourier series coefficient multiplying $\sin(\pi t/T)$ in (23) equals $\sin(\varphi)$, but in our approximation $x_r(t)=2\cos(\varphi)\cos(\pi t/T)$ is identically equal to zero. Thus, we can say that it is evidently underestimated.

Comparison of the three cases regarding possible choice of the coefficient *c*, which were discussed above, shows that the best of them is the first one with $c=1/2$. Why? Because this choice assures a correct calculation (reconstruction) of two from a total number of three Fourier series coefficients of the periodic signal given by (6).

4 CONCLUSIONS

It is well known that a critical sampling of an analog signal can lead to ambiguous results in the sense that the reconstructed signal is not unique. Such is the case of sampling of a cosinusoidal signal of any phase considered in very detail in this paper.

The non-unique results obtained for this case as well as the reasons of a lack of uniqueness are thoroughly explained here and in an accompanying paper (Borys A., Korohoda P. 2020). Furthermore, it is also shown that manipulating values of the transfer function of an ideal rectangular reconstruction filter at the transition edges does not eliminate the ambiguity incorporated in the result of signal reconstruction achieved.

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