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Global Stability of Fractional Feedback Systems with Positive Linear Parts

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ABSTRACT: The global (absolute) stability of fractional nonlinear systems with negative feedbacks and positive not necessary asymptotically stable linear parts is addressed. It is shown that the coefficients of the transfer matrix of fractional positive asymptotically stable systems are positive. Sufficient conditions for the global stability of the fractional nonlinear systems with positive linear parts are established.

1 INTRODUCTION

In positive systems inputs, state variables and outputs take only nonnegative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollutions models. A variety of models having positive behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc. Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of the art in positive systems theory is given in the monographs [1, 4, 8].

Positive linear systems with different fractional orders have been addressed in [9, 10]. Stability of standard and positive systems has been investigated in [5, 15, 17, 20] and of fractional systems in [3, 6, 13, 14]. Descriptor positive systems have been analyzed in [11, 12]. Linear positive electrical circuits with state feedbacks have been addressed in [2, 15].The global stability of the nonlinear systems with positive linear parts has been analyzed in [7].

In this paper the main results of [7] will be extended to fractional nonlinear systems and the global stability of fractional nonlinear systems with negative feedbacks and positive not necessary asymptotically stable linear parts will be addressed.

The paper is organized as follows. In section 2 some preliminaries concerning positive linear systems are given and it is shown that the coefficients of the transfer matrices of positive asymptotically stable linear systems are positive. The main result of the paper is given in section 3 where the sufficient conditions for the global stability of the fractional nonlinear feedback systems with positive linear parts are established. Concluding remarks are given in section 4.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ - the set of $n\times m$ real matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, M_n - the set of $n\times m$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n - the $n\times m$ identity matrix.

2 PRELIMINARIES

Consider the continuous-time linear system

$$\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + Bu(t), \ 0 < \alpha < 1 \tag{1a}$$

$$y(t) = Cx(t) + Du(t) \tag{1b}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$, $y(t) \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$, $D \in \Re^{p\times m}$,

$$\frac{d^\alpha x(t)}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{x}(\tau)}{(t-\tau)^\alpha} d\tau, \ \dot{x}(\tau) = \frac{dx(\tau)}{d\tau} \tag{1c}$$

is the Caputo fractional derivative and

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \ \mathrm{Re}(z) > 0 \tag{1d}$$

is the gamma function [13].

Definition 1. [7] The fractional system (1) is called (internally) positive if $x(t) \in \Re_+^n$ and $y(t) \in \Re_+^p$, $t \ge 0$ for any initial conditions $x(0) \in \Re_+^n$ and all inputs $u(t) \in \Re_+^m$, $t \ge 0$.

Theorem 1. [7] The fractional system (1) is positive if and only if

$$A \in M_n, B \in \Re_+^{n\times m}, C \in \Re_+^{p\times n}, D \in \Re_+^{p\times m} \tag{2}$$

Definition 2. [6, 12] The positive fractional system (1) (for $u(t)$=0) is called asymptotically stable if

$$\lim_{t\to\infty} x(t) = 0 \ \text{ for any } \ x(0) \in \Re_+^n . \tag{3}$$

Theorem 2. [6, 12] The positive linear system (1) (for $u(t)$=0) is asymptotically stable if and only if one of the following equivalent conditions is satisfied:

1. All coefficient of the characteristic polynomial

$$p_n(s) = \det[I_n s - A] = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 \tag{4}$$

are positive, i.e. $a_i > 0$ for $i = 0,1,\ldots,n-1$.

2. There exists strictly positive vector $\lambda^T = [\lambda_1 \ \cdots \ \lambda_n]^T$, $\lambda_k > 0$, $k = 1,\ldots,n$ such that

$$A\lambda < 0 \ \text{ or } \ \lambda^T A < 0 . \tag{5}$$

The transfer matrix of the system (1) is given by

$$T(s) = C[I_n\bar{\lambda} - A]^{-1}B + D, \ \bar{\lambda} = s^\alpha . \tag{6}$$

Theorem 3. If the matrix $A \in M_n$ is Hurwitz and $B \in \Re_+^{n\times m}$, $C \in \Re_+^{p\times n}$, $D \in \Re_+^{p\times m}$ of the linear positive system (1), then all coefficients of the transfer matrix (3) are positive.

Proof is similar to the proof given in [7] for the standard positive linear systems..

Example 1. Consider the fractional positive linear system (1) with the matrices

$$A = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, C = [1 \ \ 1], D = [2], \tag{7}$$

Note that the matrix A given by (7) is Hurwitz since its characteristic polynomial

$$\det\left[I_2\bar{\lambda} - A\right] = \begin{vmatrix} \bar{\lambda}+2 & -1 \\ -2 & \bar{\lambda}+3 \end{vmatrix} = \bar{\lambda}^2 + 5\bar{\lambda} + 4 \tag{8}$$

has positive coefficients (Theorem 2).

Using (7) and (6) we obtain

$$T(\bar{\lambda}) = C\left[I_n\bar{\lambda} - A\right]^{-1} B + D = \\ [1 \ \ 1]\begin{bmatrix} \bar{\lambda}+2 & -1 \\ -2 & \bar{\lambda}+3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + [2] = \frac{2\bar{\lambda}^2 + 13\bar{\lambda} + 19}{\bar{\lambda}^2 + 5\bar{\lambda} + 4} \tag{9}$$

The transfer function (9) has positive coefficients.

3 MAIN RESULT

Consider the nonlinear feedback system shown in Figure 1 consisting of the fractional linear part described by the equations

$$\frac{d^\alpha x(t)}{dt^\alpha} = Ax + bu, \tag{10a}$$

$$y = cx, \tag{10b}$$

where $x = x(t) \in \Re^n$, $u = u(t) \in \Re$, $y = y(t) \in \Re$, $A \in \Re^{n\times n}$, $b \in \Re^n$, $c \in \Re^{1\times n}$ and of the nonlinear element with the characteristic $u=f(e)$ shown in Figure 2.

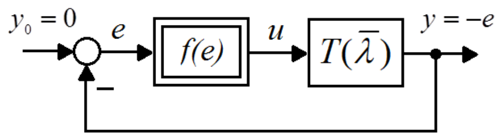

Figure 1. Nonlinear feedback system.

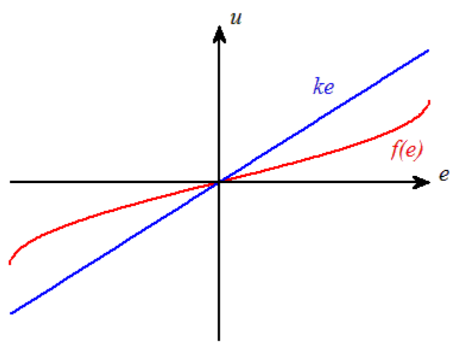

Figure 2. Characteristic of the nonlinear element.

The characteristic of the nonlinear element satisfies the condition

$$f(0) = 0 \ \text{ and } \ 0 \le \frac{f(e)}{e} \le k, \ \ k < +\infty . \tag{12}$$

It is assumed that the linear part (10) is positive, i.e.

$$A \in M_n,\ \ b \in \Re_+^n,\ \ c \in \Re_+^{1\times n}, \tag{13}$$

but not necessary asymptotically stable.

It is also assumed that if the linear part is unstable then by suitable choice of the gain k_1 we may obtain (Figure 3) asymptotically stable positive linear part with the transfer function

$$T_1(\bar{\lambda}) = \frac{T(\bar{\lambda})}{1 + k_1 T(\bar{\lambda})} \tag{14}$$

and the nonlinear element with the characteristic $f_1(e) = f(e) - k_1 e$ satisfies the condition (Figure 4)

$$f_1(0),\ k_1 \le \frac{f_1(e)}{e} \le k_2 = k - k_1. \tag{15}$$

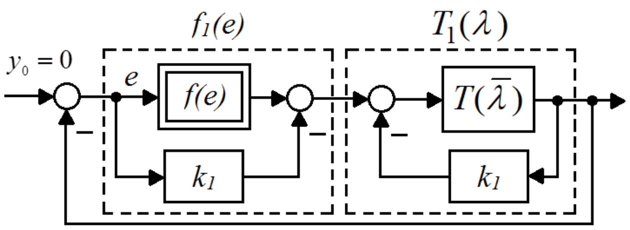

Figure 3. Nonlinear feedback system with the gain k_1.

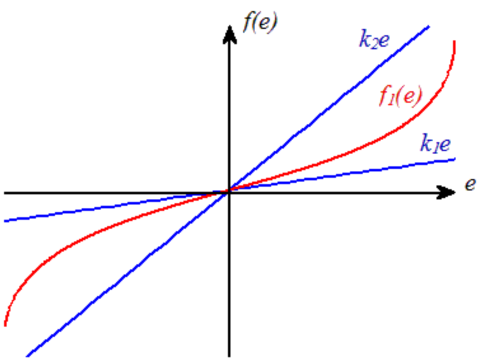

Figure 4. Characteristic of the nonlinear element with the gain k_1.

Definition 3. The nonlinear system is called globally (or absolutely) asymptotically stable if $\lim_{t\to\infty} x(t) = 0$ for any $x(0) \in \Re_+^n$.

Definition 4. The circle in the plane $(P(\omega), Q(\omega))$ with center in the point $\left(-\frac{k_1 + k_2}{2k_1k_2}, 0\right)$ and radius $\frac{k_2 - k_1}{2k_1k_2}$ will be called the $\left(-\frac{1}{k_1}, -\frac{1}{k_2}\right)$ circle.

Theorem 4. The fractional nonlinear feedback system (Figure 3) consisting of positive linear asymptotically stable part with the transfer function (14) and of nonlinear element with characteristic satisfying the condition (15) is globally asymptotically stable if the Nyquist plot of $T_1(j\omega) = P(\omega) + jQ(\omega)$ of the linear part is located on the right-hand side of the circle $\left(-\frac{1}{k_1}, -\frac{1}{k_2}\right)$.

Proof. Proof is based on the application of the Lyapunov method to the positive nonlinear system [12, 15, 17]. As the Lyapunov function we choose the time function

$$V(t) = \lambda^T e^{A_1 t} b > 0,\ t \in [0, +\infty), \tag{16}$$

where $\lambda^T = [\lambda_1\ \cdots\ \lambda_n]^T$ is strictly positive vector, i.e. $\lambda_k > 0$, $k = 1,\dots,n$.

The function $V(t) > 0$ for $t \in [0, +\infty)$ since $A_1 \in M_n$ is asymptotically stable and $b \in \Re_+^n$.

From (16) we have

$$\dot{V}(t) = \frac{dV(t)}{dt} = \lambda^T A_1 e^{A_1 t} b < 0 \quad \text{for} \quad t \in [0, +\infty) \tag{17}$$

since $\lambda^T A_1 < 0$ for the Hurwitz matrix $A_1 \in M_n$ (Theorem 2).

Therefore, by the Lyapunov theorem the fractional positive nonlinear system is asymptotically stable if

$$c e^{A_1 t} b > 0 \quad \text{for} \quad t \in [0, +\infty). \tag{18}$$

Note that

$$T_1(\bar{\lambda}) = c\mathcal{L}[e^{A_1 t}]b = c[I_n \bar{\lambda} - A_1]^{-1} b, \tag{19}$$

where $\mathcal{L}$ is the Laplace transform operator. From (18) we obtain

$$\operatorname{Re} T_1(j\omega) + \frac{1}{k} > 0 \quad \text{for} \quad \omega \ge 0 \quad \text{and} \quad k = k_2 - k_1 > 0. \tag{20}$$

Taking into account that

$$\operatorname{Re} T_1(j\omega) + \frac{1}{k_2 - k_1} = \operatorname{Re}\left[\frac{T(j\omega)}{1 + k_1 T(j\omega)} + \frac{1}{k_2 - k_1}\right] = \frac{1}{k_2 - k_1} \operatorname{Re}\left[\frac{1 + k_2 T(j\omega)}{1 + k_1 T(j\omega)}\right] \tag{21}$$

and that the border of asymptotic stability is the $j\omega$ axis we obtain

$$j\omega = \frac{1 + k_2[P(\omega) + jQ(\omega)]}{1 + k_1[P(\omega) + jQ(\omega)]} \tag{22a}$$

or

$$j\omega\{1 + k_1[P(\omega) + jQ(\omega)]\} = 1 + k_2[P(\omega) + jQ(\omega)]. \tag{22b}$$

From (22b) we have

$$-\omega k_1 Q(\omega) = 1 + k_2 P(\omega) \quad \text{and} \quad \omega[1 + k_1 P(\omega)] = k_2 Q(\omega) \tag{23}$$

and after elimination of ω

$$[1 + k_1 P(\omega)][1 + k_2 P(\omega)] + k_1 k_2 Q^2(\omega) = 0 \tag{24a}$$

or

$$\frac{1}{k_1 k_2} + \frac{k_1 + k_2}{k_1 k_2} P(\omega) + P^2(\omega) + Q^2(\omega) = 0. \tag{24b}$$

Note that (24b) can be rewritten in the form of the equation

$$\left[P(\omega)+\frac{k_1+k_2}{2k_1k_2}\right]^2+Q^2(\omega)=\left(\frac{k_2-k_1}{2k_1k_2}\right)^2 \tag{25}$$

which describes the circle $\left(-\frac{1}{k_1},-\frac{1}{k_2}\right)$ (see Figure 5). This completes the proof. □

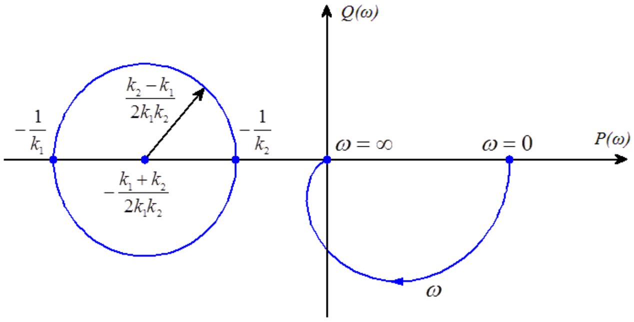

Figure 5. Nyquist plot with the circle $\left(-\frac{1}{k_1},-\frac{1}{k_2}\right)$.

This theorem can be considered as an extension for the fractional nonlinear systems with positive linear parts of the Kudrewicz theorem presented in [18] for nonlinear systems with standard linear parts.

Example 2. Consider the fractional nonlinear system with unstable linear part with

$$T(\bar{\lambda})=\frac{L(\bar{\lambda})}{M(\bar{\lambda})}=\frac{2\bar{\lambda}+3}{\bar{\lambda}^2+1.8\bar{\lambda}-0.1} \tag{26}$$

and nonlinear element with the characteristic $u=f(e)$ shown in Figure 6.

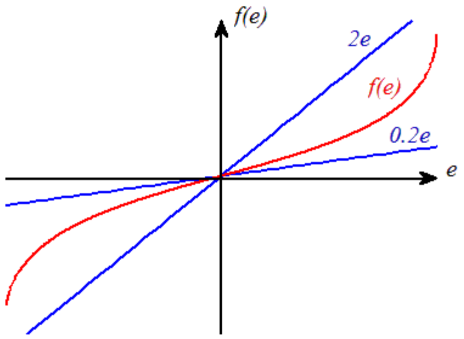

Figure 6. Characteristic of the nonlinear element of Example 2.

To obtain the fractional nonlinear system with asymptotically stable linear part we choose $k_1=0.2$ and we obtain

$$T_1(\bar{\lambda})=\frac{T(\bar{\lambda})}{1+k_1T(\bar{\lambda})}=\frac{L(\bar{\lambda})}{M(\bar{\lambda})+k_1L(\bar{\lambda})}= \frac{2\bar{\lambda}+3}{\bar{\lambda}^2+1.8\bar{\lambda}-0.1+0.2(2\bar{\lambda}+3)}=\frac{2\bar{\lambda}+3}{\bar{\lambda}^2+2.2\bar{\lambda}+0.5}. \tag{27}$$

Note that the characteristic of the nonlinear element $u=f(e)$ satisfies the condition (Figure 6)

$$0.2<\frac{f(e)}{e}<2. \tag{28}$$

In this case

$$T_1(j\omega)=\frac{j2\omega+3}{0.5-\omega^2+j2.2\omega}=P(\omega)+jQ(\omega), \tag{29}$$

where

$$P(\omega)=\frac{1.4\omega^2+1.5}{(0.5-\omega^2)^2+(2.2\omega)^2},\ Q(\omega)=-\frac{2\omega^3+5.6\omega}{(0.5-\omega^2)^2+(2.2\omega)^2} \tag{30}$$

The Nyquist plot and the circle are shown on the Figure 7. By Theorem 4 the nonlinear system is globally stable.

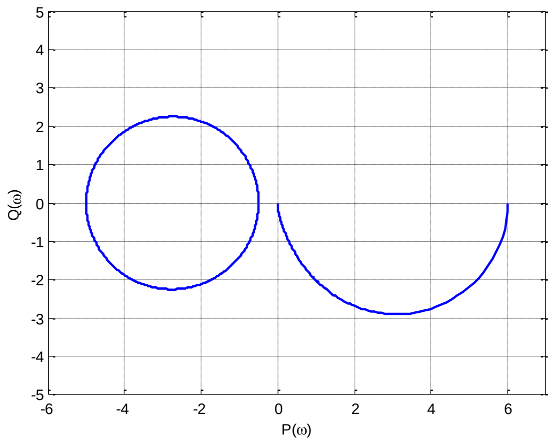

Figure 7. Nyquist plot with the circle (-5,-0.5).

4 CONCLUDING REMARKS

The global stability of fractional nonlinear systems with negative feedbacks and positive linear parts has been analyzed. The characteristics $u=f(e)$ of the nonlinear element satisfy the assumption (12) and the linear parts described by the equations (11) are not necessary asymptotically stable. The gain k_1 of the positive linear part has been chosen so that the transfer function (14) is asymptotically stable and the characteristic $u=f(e)$ satisfies the condition (15).

It has been shown that the nonlinear systems are globally asymptotically stable if the Nyquist plots of the linear parts are located on the right-hand side of the circles $\left(-\frac{1}{k_1},-\frac{1}{k_2}\right)$. This theorem is an extension of the Kudrewicz theorem presented in [18] for nonlinear systems with standard linear parts.

The considerations have been illustrated by numerical examples. The considerations can be extended to the fractional nonlinear systems with positive linear parts and with positive descriptor linear parts.

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