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Global Stability of Different Fractional Orders Multi Input Multi Output Nonlinear Feedback Systems with Interval Matrices of Positive Linear Parts

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ABSTRACT: The global stability of continuous-time fractional orders multi–input multi-output nonlinear feedback systems with interval matrices of positive linear parts is investigated. New sufficient conditions for the global stability of these class of positive nonlinear systems are established. A procedure for computation of gain matrix characterizing the class of nonlinear elements is given and illustrated on simple example.

1 INTRODUCTION

In positive systems inputs, state variables and outputs take only nonnegative values for any nonnegative inputs and nonnegative initial conditions [4, 1, 9]. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollutions models. A variety of models having positive behaviour can be found in engineering, management science, economics, social sciences, biology and medicine, etc. An overview of state of the art in positive systems theory is given in the monographs [1, 4, 9, 13, 17].

Mathematical fundamentals of the fractional calculus are given in the monographs [21, 22, 13, 17]. The positive fractional linear systems have been investigated in [3, 5, 7, 10-14, 17, 20, 23-25]. Positive linear systems with different fractional orders have been addressed in [10, 11, 27]. Descriptor positive systems have been analyzed in [2, 12] and their stabilization in [26, 27]. Linear positive electrical circuits with state feedbacks have been addressed in [2, 17]. The superstabilization of positive linear electrical circuits by state feedbacks have been analyzed in [15] and the stability of nonlinear systems in [16, 18]. The global stability of nonlinear systems with negative feedbacks and positive not necessary asymptotically stable linear parts has been investigated in [6, 8]. The global stability of nonlinear standard and fractional positive feedback systems has been considered in [15].

In this paper the global stability of nonlinear fractional orders feedback multi-input multi-output systems with interval matrices of positive linear parts will be addressed.

The paper is organized as follows. In section 2 the basic definitions and theorems concerning the positive different fractional orders linear systems are recalled. The stability of fractional interval positive linear systems is analyzed in section 3. New sufficient conditions for the global stability of these feedback nonlinear systems with interval matrices of positive linear parts are established in section 4. In section 5 a procedure for calculation of gain matrix characterizing the class of nonlinear element is presented and illustrated by numerical examples. Concluding remarks are given in section 6.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices,

$\Re_+^{n\times m}$ - the set of $n\times m$ real matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, M_n - the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n - the $n\times n$ identity matrix, $\sum A(:,n)$ - sum of all elements of nth column.

2 POSITIVE DIFFERENT FRACTIONAL ORDERS LINEAR SYSTEMS

Consider the fractional continuous-time linear system

$$\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + Bu(t), 0 < \alpha < 1 \tag{1a}$$

$$y(t) = Cx(t) + Du(t), \tag{1b}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$, $y(t) \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$, $D \in \Re^{p\times m}$,

$$\frac{d^\alpha x(t)}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{x}(\tau)}{(t-\tau)^\alpha} d\tau, \dot{x}(\tau) = \frac{dx(\tau)}{d\tau} \tag{1c}$$

is the Caputo fractional derivative and

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \mathrm{Re}(z) > 0 \tag{1d}$$

is the gamma function [13, 18].

Definition 1. [13, 18] The fractional system (1) is called (internally) positive if $x(t) \in \Re_+^n$ and $y(t) \in \Re_+^p$, $t \ge 0$ for any initial conditions $x(0) \in \Re_+^n$ and all inputs $u(t) \in \Re_+^m$, $t \ge 0$.

Theorem 1. [13, 18] The fractional system (1) is positive if and only if

$$A \in M_n, B \in \Re_+^{n\times m}, C \in \Re_+^{p\times n}, D \in \Re_+^{p\times m} \tag{2}$$

Definition 2. [13, 18] The positive fractional system (1) (for $u(t)$=0) is called asymptotically stable (the matrix A is Hurwitz) if

$$\lim_{t\to\infty} x(t) = 0 \text{ for any } x(0) \in \Re_+^n. \tag{3}$$

Theorem 2. [13, 18] The fractional positive linear system (1) is asymptotically stable if and only if one of the following equivalent conditions is satisfied:

1. All coefficient of the characteristic polynomial

$$p_n(s) = \det[I_n s - A] = s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0 \tag{4}$$

are positive, i.e. $a_i > 0$ for $i = 0,1,\dots,n-1$.

2. There exists strictly positive vector $\lambda^T = [\lambda_1 \ \cdots \ \lambda_n]^T$, $\lambda_k > 0$, $k = 1,\dots,n$ such that

$$A\lambda < 0 \text{ or } \lambda^T A < 0. \tag{5}$$

Theorem 3. The fractional positive system (1) is asymptotically stable if the sum of entries of each column (row) of the matrix A is negative.

Proof. Using (5) we obtain

$$A\lambda = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \dots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} = \begin{bmatrix} a_{11} \\ \vdots \\ a_{n1} \end{bmatrix} \lambda_1 + \dots + \begin{bmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{bmatrix} \lambda_n < \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \tag{6}$$

and the sum of entries of each column of the matrix A is negative since $\lambda_k > 0$, $k = 1,\dots,n$. The proof for rows is similar. □

Now consider the fractional linear system with two different fractional orders

$$\begin{bmatrix} \frac{d^\alpha x_1(t)}{dt^\alpha} \\ \frac{d^\beta x_2(t)}{dt^\beta} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t), \tag{7a}$$

$$y(t) = [C_1 \quad C_2] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \tag{7b}$$

where $0 < \alpha, \beta < 1$, $x_1(t) \in \Re^{n_1}$ and $x_2(t) \in \Re^{n_2}$ are the state vectors, $A_{ij} \in \Re^{n_i \times n_j}$, $B_i \in \Re^{n_i \times m}$, $C_i \in \Re^{p\times n_i}$; i, j = 1,2; $u(t) \in \Re^m$ is the input vector and $y(t) \in \Re^p$ is the output vector. Initial conditions for (7a) have the form

$$x_1(0) = x_{10}, x_2(0) = x_{20} \text{ and } x_0 = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}. \tag{7c}$$

Remark 1. The state equation (7a) of fractional continuous-time linear systems with two different fractional orders has similar structure as the 2D Roeesser type models.

Definition 3. The fractional system (7) is called positive if $x_1(t) \in \Re^{n_1}$ and $x_2(t) \in \Re^{n_2}$ and $y(t) \in \Re_+^p$, $t \ge 0$ for any initial conditions $x_{10} \in \Re_+^{n_1}$, $x_{20} \in \Re_+^{n_2}$ and all input vectors $u \in \Re_+^m$, $t \ge 0$.

Theorem 4. [10, 11] The fractional system (7) for $0 < \alpha < 1$; $0 < \beta < 1$ is positive if and only if

$$\tilde{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in M_n, \tilde{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \in \Re_+^{n\times m}, \\ \tilde{C} = [C_1 \quad C_2] \in \Re_+^{p\times n} \ (n = n_1 + n_2) \tag{8}$$

Theorem 5. [13] The fractional positive system (7) is asymptotically stable if and only if one of the following equivalent conditions is satisfied:

1. All coefficients of the characteristic polynomial

$$p_n(s^\alpha, s^\beta) = \det\left[\begin{bmatrix} I_{n_1}s^\alpha & 0 \\ 0 & I_{n_2}s^\beta \end{bmatrix} - \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\right] \tag{9}$$

are positive.

2. There exists strictly positive vector $\lambda = [\lambda_1 \ \cdots \ \lambda_n]$, $\lambda_k > 0$, $k = 1,\dots,n$ such that

$$\tilde{A}\lambda < 0 \text{ or } \lambda^T \tilde{A} < 0. \tag{10}$$

Theorem 6. The fractional positive system (7) is asymptotically stable if the sum of entries of each column (row) if the matrix $\tilde{A}$ is negative.

Proof is similar to the proof of Theorem 3.

Theorem 7. The solution of the equation (7a) for $0<\alpha<1;\ 0<\beta<1$ with initial conditions (7c) has the form

$$x(t)=\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}=\Phi_0(t)x_0+\int_0^t M(t-\tau)u(\tau)d\tau\,, \tag{11}$$

where

$$\begin{aligned} M(t)&=\Phi_1(t)B_{10}+\Phi_2(t)B_{01}= \\ &=\begin{bmatrix} \Phi_{11}^1(t) & \Phi_{12}^1(t) \\ \Phi_{21}^1(t) & \Phi_{22}^1(t) \end{bmatrix}\begin{bmatrix} B_1 \\ 0 \end{bmatrix}+\begin{bmatrix} \Phi_{11}^2(t) & \Phi_{12}^2(t) \\ \Phi_{21}^2(t) & \Phi_{22}^2(t) \end{bmatrix}\begin{bmatrix} 0 \\ B_2 \end{bmatrix}= \\ &=\begin{bmatrix} \Phi_{11}^1(t)B_1+\Phi_{12}^2(t)B_2 \\ \Phi_{21}^1(t)B_1+\Phi_{22}^2(t)B_2 \end{bmatrix}=\begin{bmatrix} \Phi_{11}^1(t) & \Phi_{12}^2(t) \\ \Phi_{21}^1(t) & \Phi_{22}^2(t) \end{bmatrix}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \end{aligned} \tag{12a}$$

and

$$\Phi_0(t)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}T_{kl}\frac{t^{k\alpha+l\beta}}{\Gamma(k\alpha+l\beta+1)}, \tag{12b}$$

$$\Phi_1(t)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}T_{kl}\frac{t^{(k+1)\alpha+l\beta-1}}{\Gamma\left[(k+1)\alpha+l\beta\right]}, \tag{12c}$$

$$\Phi_2(t)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}T_{kl}\frac{t^{k\alpha+(l+1)\beta-1}}{\Gamma\left[k\alpha+(l+1)\beta\right]}, \tag{12d}$$

$$T_{kl}=\begin{cases} I_n & \text{for} \quad k=l=0 \\ \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} & \text{for} \quad k=1, l=0 \\ \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix} & \text{for} \quad k=0, l=1 \\ T_{10}T_{k-1,l}+T_{01}T_{k,l-1} & \text{for} \quad k+l>1 \end{cases} \tag{12e}$$

Proof is given in [11].

Note that if $\alpha=\beta$ then from (12b) we have

$$\Phi_0\big|_{\alpha=\beta}(t)=\sum_{k=0}^{\infty}\frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)}. \tag{13}$$

3 STABILITY OF FRACTIONAL INTERVAL POSITIVE LINEAR SYSTEMS

Consider the fractional interval positive linear continuous-time system

$$\frac{d^\alpha x}{dt^\alpha}=Ax,\ 0<\alpha<1, \tag{14}$$

where $x=x(t)\in\Re^n$ is the state vector and the interval matrix $A\in M_n$ is defined by

$$\underline{A}\le A\le\bar{A} \quad \text{or equivalently} \quad A\in[\bar{A},\underline{A}]. \tag{15}$$

Definition 4. The fractional interval positive system (14) is called asymptotically stable if the system is asymptotically stable for all matrices $A\in M_n$ satisfying the condition (15).

By condition (5) of Theorem 2 the positive system (14) is asymptotically stable if there exists strictly positive vector $\lambda>0$ such that the condition (5) is satisfied.

For two fractional positive linear systems

$$\frac{d^\alpha x}{dt^\alpha}=\bar{A}x,\ \bar{A}\in M_n \tag{16a}$$

and

$$\frac{d^\alpha x}{dt^\alpha}=\underline{A}x,\ \underline{A}\in M_n \tag{16b}$$

there exists a strictly positive vector $\lambda\in\Re_+^n$ such that

$$\bar{A}\lambda<0 \quad \text{and} \quad \underline{A}\lambda<0 \tag{17}$$

if and only if the systems (16) are asymptotically stable.

Remark 2. As according to condition 2 of Theorem 2 (Theorem 5), in general case, it is not necessary to find the same $\lambda\in\Re_+^n$ vector for two different systems. However, for interval systems, it is reasonable for choose one $\lambda\in\Re_+^n$ vector for (16a) and (16b).

Theorem 8. If the matrices $\bar{A}$ and $\underline{A}$ of fractional positive systems (16) are asymptotically stable then their convex linear combination

$$A=(1-k)\underline{A}+k\bar{A} \quad \text{for} \quad 0\le k\le 1 \tag{18}$$

is also asymptotically stable.

Proof. By condition (5) of Theorem 2 if the fractional positive linear systems (16) are asymptotically stable then there exists strictly positive vector $\lambda\in\Re_+^n$ such that (17) holds. Using (5) and (17) we obtain

$$A\lambda=[(1-k)\underline{A}+k\bar{A}]\lambda=(1-k)\underline{A}\lambda+k\bar{A}\lambda<0 \tag{19}$$

for $0\le k\le 1$. Therefore, if the positive linear systems (16) are asymptotically stable then their convex linear combination (18) is also asymptotically stable. □

Theorem 9. The interval positive system (14) is asymptotically stable if and only if the positive linear systems (16) are asymptotically stable.

Proof. By condition (5) of Theorem 2, if the matrices $\bar{A}\in M_n$, $\underline{A}\in M_n$ are asymptotically stable, then there exists a strictly positive vector $\lambda\in\Re_+^n$ such that (5) holds. The convex linear combination (18) satisfies the condition $A\lambda<0$ if and only if (19) holds. Therefore, the interval system (14) is asymptotically stable if and only if the positive linear system is asymptotically stable. □

Example 1. Consider the fractional interval positive linear continuous-time system (14) with the matrices

$$\bar{A}=\begin{bmatrix} -3 & 2 \\ 2 & -4 \end{bmatrix},\ \underline{A}=\begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix}. \tag{20}$$

Using the condition (5) of Theorem 2 we choose $\lambda = [1\ \ 1]^T$ and we obtain

$$\bar{A}\lambda = \begin{bmatrix} -3 & 2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} < 0 \tag{21a}$$

and

$$\underline{A}\lambda = \begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} < 0. \tag{21b}$$

Therefore, the matrices (20) are Hurwitz.

These considerations can be easily extended to positive different fractional orders linear systems (7).

4 GLOBAL STABILITY OF FRACTIONAL NONLINEAR POSITIVE FEEDBACK SYSTEMS

Consider the m-input p-output (MIMO) nonlinear feedback system shown in Figure 1 which consists of the positive fractional linear part, the nonlinear element with the matrix characteristic $u=f(e)$ and the feedback with positive gain matrix H. The positive fractional linear part is described by the equations (7) with the interval matrices

$$\tilde{A} \in [\bar{A}, \underline{A}] \in M_n,\ \tilde{B} \in [\bar{B}, \underline{B}] \in \Re_+^{n\times m},\ \tilde{C} \in [\bar{C}, \underline{C}] \in \Re_+^{p\times n}. \tag{22}$$

It is assumed that the interval matrix $\tilde{A} \in M_n$ is Hurwitz.

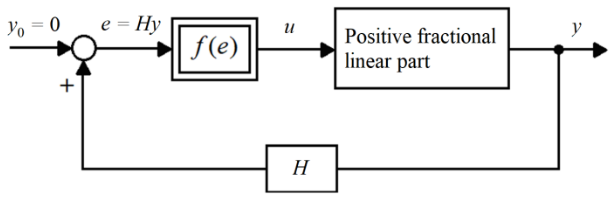

Figure 1. The nonlinear feedback system

The characteristic $f(e)$ of the nonlinear element satisfies the condition

$$0 \le f(e) \le Ke \quad \text{or} \quad u \le Ke, \tag{23a}$$

where

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}, K = \begin{bmatrix} k_{11} & \cdots & k_{1p} \\ \vdots & \ddots & \vdots \\ k_{m1} & \cdots & k_{mp} \end{bmatrix}, e = \begin{bmatrix} e_1 \\ \vdots \\ e_p \end{bmatrix} \tag{23b}$$

and

$$f(0) = 0,\ u_i = f(e_1,\dots,e_p) \le k_{i1}e_1 + \dots + k_{ip}e_p,\ i = 1,\dots,m \tag{23c}$$

Remark 3. The matrix K forms m, p-dimensional convex surfaces which constricts possible dynamics of nonlinear elements, i.e. for $m = p = 1$ the possible trajectory $f(e)$ of nonlinear element should fit in grey cones presented in the Figure 2.

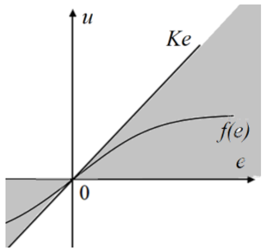

Figure 2. Example of possible trajectory $f(e)$ of nonlinear element constricted by K.

Definition 5. The fractional nonlinear positive system is called globally stable if it is asymptotically stable for all nonnegative initial conditions $x(0) \in \Re_+^n$.

The following theorem gives sufficient conditions for the global stability of the positive nonlinear system.

Theorem 10. The fractional nonlinear system consisting of the positive asymptotically stable linear part described by (7) with interval matrices (22), the nonlinear element satisfying the condition (23) and the feedback with positive gain matrix $H \in \Re_+^{m\times p}$ is globally stable if, the sum of entries of each column (row) of the matrix

$$(1-q)\underline{A} + q\bar{A} + \tilde{B}KH\tilde{C} = \begin{cases} \bar{A} + \bar{B}\bar{K}H\bar{C} \in M_n & \text{for} \quad q = 0 \\ \underline{A} + \underline{B}\underline{K}H\underline{C} \in M_n & \text{for} \quad q = 1 \end{cases} \tag{24}$$

are negative.

Proof. The proof will be accomplished by the use of the Lyapunov method [19, 20]. As the Lyapunov function $V(x)$ we choose

$$V_1(x_1) + V_2(x_2) = \lambda_1^T x_1 + \lambda_2^T x_2 \ge 0 \quad \text{for}$$
$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \Re_+^n,\ \lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \in \Re_+^n, \tag{25}$$

where λ is strictly positive vector, i.e. $\lambda_{ij} > 0$, $i = 1,2;\ j = 1,\dots,n.$

Using (25) and (22) we obtain

$$\frac{d^\alpha V_1(x_1)}{dt^\alpha} + \frac{d^\beta V_2(x_2)}{dt^\beta} = [\lambda_1^T \quad \lambda_2^T] \begin{bmatrix} \dfrac{d^\alpha x_1}{dt^\alpha} \\ \dfrac{d^\beta x_2}{dt^\beta} \end{bmatrix} = \tag{26}$$
$$\lambda^T(\tilde{A}x + \tilde{B}u) = \lambda^T(\tilde{A}x + \tilde{B}f(e)) \le \lambda^T(\tilde{A} + \tilde{B}KH\tilde{C})x$$

since $u = f(e) \le Ke = KH\tilde{C}x.$

From (26) it follows that $\dfrac{d^\alpha V_1(x_1)}{dt^\alpha} + \dfrac{d^\beta V_2(x_2)}{dt^\beta} < 0$ if the sum of entries of each column (row) of the matrix (24) is negative (Theorem 3) and the nonlinear positive system is globally stable. □

5 PROCEDURE AND EXAMPLE

To find the $K \in \Re_+^{m\times p}$ satisfying the condition (24) for the nonlinear positive system the following procedure can be used.

Procedure 1.

Step 1. Using the matrices $\bar{A}, \bar{B}, \bar{C}$ of the positive linear system and the matrix H compute the maximum value of the matrix $\bar{K}$ such that the sum of all entries of each column (row) of the matrix

$$\bar{\bar{A}} = \bar{A} + \bar{B}\bar{K}H\bar{C} \tag{27}$$

is negative.

Entries of the matrix K can be compute as the solution of the linear matrix equation

$$Gk = h, \tag{28}$$

where the matrix G and the column vector h are defined by the sum of entries of each column (row) of the matrix (27) and vector k contains components of matrix K. If $mp > n$ or/and $\operatorname{rank} G \neq n$, then we choose arbitrarily $mp - \operatorname{rank} G$ nonnegative entries of the matrix K.

Step 2. Using the matrices $\underline{A}, \underline{B}, \underline{C}$ of the positive linear system and the matrix H compute the maximum value of the matrix $\underline{K}$ such that the sum of all entries of each column (row) of the matrix

$$\underline{\underline{A}} = \underline{A} + \underline{B}\underline{K}H\underline{C} \tag{29}$$

is negative.

Step 3. Taking into considerations $\bar{K}$ computed in Step 1 and $\underline{K}$ from Step 2, find the desired $K \in \Re_+^{m\times p}$ for which, the matrices $\bar{A}$ and $\underline{A}$ are Hurwitz, i.e. the characteristic $f(e)$ satisfy the condition (23a).

Remark 4. The conditions of Theorem 2 can be also used to compute the entries of the matrix K. Usually in this case the computations are more complicated.

Example 2. Consider the nonlinear feedback system shown in Figure 1 with the interval matrices of the positive linear part

$$\bar{A} = \begin{bmatrix} -15.8 & 0.2 & 0.9 \\ 0.4 & -14.1 & 0.5 \\ 0.5 & 0.4 & -20.1 \end{bmatrix}, \quad \underline{A} = \begin{bmatrix} -7.9 & 0.1 & 0.7 \\ 0.2 & -7.9 & 0.3 \\ 0.1 & 0.2 & -12.1 \end{bmatrix},$$
$$\bar{B} = \begin{bmatrix} 0.5 & 0.4 \\ 0.3 & 0.5 \\ 0.8 & 0.6 \end{bmatrix}, \quad \underline{B} = \begin{bmatrix} 0.4 & 0.3 \\ 0.2 & 0.4 \\ 0.6 & 0.5 \end{bmatrix}, \tag{30}$$
$$\bar{C} = \begin{bmatrix} 0.4 & 0.6 & 0.8 \\ 0.6 & 0.4 & 0.6 \end{bmatrix}, \quad \underline{C} = \begin{bmatrix} 0.3 & 0.5 & 0.6 \\ 0.4 & 0.3 & 0.5 \end{bmatrix}$$

and the gain matrix

$$H = \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}. \tag{31}$$

From (30) it follows that $m = p = 2$ and $u_1 = f(e_1, e_2) \le k_{11}e_1 + k_{12}e_2$, $u_2 = f(e_1, e_2) \le k_{21}e_1 + k_{22}e_2$, then we are looking for $K = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \in \Re_+^{2\times 2}$, for which the nonlinear feedback system is globally stable.

Using Procedure 1 we obtain:

Step 1. Using (27), (30) and (31) we obtain that the sum of columns are the following:

$$\begin{aligned} &\sum \bar{\bar{A}}(:,1) = 2.24k_{11} + 5.44k_{12} + 2.1k_{21} + 5.1k_{22} - 14.9, \\ &\sum \bar{\bar{A}}(:,2) = 2.56k_{11} + 4.16k_{12} + 2.4k_{21} + 3.9k_{22} - 13.5, \\ &\sum \bar{\bar{A}}(:,3) = 3.52k_{11} + 6.08k_{12} + 3.3k_{21} + 5.7k_{22} - 18.7 \end{aligned} \tag{32}$$

and taking into consideration (28), we have

$$G = \begin{bmatrix} 2.24 & 5.44 & 2.1 & 5.1 \\ 2.56 & 4.15 & 2.4 & 3.9 \\ 3.52 & 6.08 & 3.3 & 5.7 \end{bmatrix}, \quad k = \begin{bmatrix} k_{11} \\ k_{12} \\ k_{21} \\ k_{22} \end{bmatrix}, \quad h = \begin{bmatrix} 14.9 \\ 13.5 \\ 18.7 \end{bmatrix}. \tag{33}$$

Since G=2, then we choose $mp - \operatorname{rank} G = 2$ elements of the vector k (entries of the matrix K) as k_{21}=1, k_{22}=1 . Solution for (28) with (33) is the following

$$\begin{bmatrix} k_{11} \\ k_{12} \\ k_{21} \\ k_{22} \end{bmatrix} = \begin{bmatrix} 1.309 \\ 0.869 \\ 1 \\ 1 \end{bmatrix}, \tag{34}$$

thus for matrix

$$\bar{K} <= \begin{bmatrix} 1.309 & 0.869 \\ 1 & 1 \end{bmatrix} \tag{35}$$

the system with $\bar{A}, \bar{B}, \bar{C}$ is stable.

Step 2. Similarly as in Step 1, using (29), (30) and (31) we can write the linear matrix equation (28) in the form

$$\begin{bmatrix} 1.1 & 2.53 & 1.2 & 2.76 \\ 1.43 & 2.2 & 1.56 & 2.4 \\ 1.87 & 3.41 & 2.04 & 3.72 \end{bmatrix} \begin{bmatrix} k_{11} \\ k_{12} \\ k_{21} \\ k_{22} \end{bmatrix} = \begin{bmatrix} 7.6 \\ 7.6 \\ 11.1 \end{bmatrix}. \tag{36}$$

Since $\operatorname{rank} \begin{bmatrix} 1.1 & 2.53 & 1.2 & 2.76 \\ 1.43 & 2.2 & 1.56 & 2.4 \\ 1.87 & 3.41 & 2.04 & 3.72 \end{bmatrix} = 2$, then we choose two elements of the vector k as k_{11}=1, k_{12}=1. Solution for (36) is the following

$$\begin{bmatrix} k_{11} \\ k_{12} \\ k_{21} \\ k_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1.013 \\ 1.002 \end{bmatrix}, \tag{37}$$

thus for matrix

$$\underline{K} <= \begin{bmatrix} 1 & 1 \\ 1.013 & 1.002 \end{bmatrix} \tag{38}$$

the system with $\underline{A}, \underline{B}, \underline{C}$ is stable.

Step 3. Taking under considerations (35) and (38), we have that for $K = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, the matrices $\bar{\bar{A}}$ and $\underline{\underline{A}}$ are Hurwitz since

$$\begin{aligned} &\sum \bar{\bar{A}}(:,1) = -0.01, \quad &&\sum \underline{\underline{A}}(:,1) = -0.02, \\ &\sum \bar{\bar{A}}(:,2) = -0.01, \ldots &&\sum \underline{\underline{A}}(:,2) = -0.48, \\ &\sum \bar{\bar{A}}(:,3) = -0.06, \quad &&\sum \underline{\underline{A}}(:,3) = -0.1. \end{aligned} \tag{39}$$

Therefore, for nonlinear element satisfying the condition

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \le f(e_1, e_2) \le \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \tag{40}$$

the nonlinear feedback system with interval matrices (30) of positive linear parts and the gain matrix (31) is globally stable.

6 CONCLUDING REMARKS

The global stability of continuous-time different fractional orders nonlinear feedback systems with interval matrices of positive linear parts has been investigated. New sufficient conditions for the global stability of this class of positive nonlinear systems are established (Theorem 10). The procedure for calculation of gain matrix characterizing the class of nonlinear element is presented and illustrated by numerical example. The considerations can be extended to discrete-time fractional different orders nonlinear systems with interval matrices of positive linear parts and scalar feedbacks. An open problem is an extension of the considerations to nonlinear different orders fractional systems with interval matrices of their positive linear parts.

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