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Dirac Delta as a Useful Technical Tool in Modelling Signals but Hard to Think About It as a Physical Signal Itself

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ABSTRACT: In this paper, we show that the Dirac delta is a useful technical tool in modelling signals but hard to think about it as a physical signal itself. This thesis is supported here by an example coming from the field of measuring physical quantities and measurement theory.

1 INTRODUCTION

In research papers and textbooks on systems theory and signal processing, in case of using a mathematical concept: the Dirac distribution (called also a Dirac delta or a Dirac impulse) in them – in different contexts – it is assumed that this distribution can be used both as an operator and as a signal (Prandoni P., Vetterli M. 2008), (Vetterli M., Kovacevic J., Goyal V. K. 2014), (Ingle K., Proakis J. G. 2012), (Oppenheim A. V., Schafer R. W., Buck J. R. 1998), (Dąbrowski A. 2008), (Howell K. B. 2001), (Gasquet C., Witomski P. 1998), (Osgood B. 2014). That is it can play a role of an operator, but also a role of a signal. And it seems that in engineering sciences, particularly in systems theory, this way of thinking has its roots in the fact that any linear (non-pathological) system can be described by the following convolution integral:

$$y(t)=\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau\,, \tag{1}$$

where *y(t)* and *x(t)* mean an output and an input signal of a system, respectively. This system is assumed to be characterized by its system's function (called also its impulse response) *h(t)*. Variable *t* in (1) stands for a continuous time.

Mathematically, (1) can be viewed as an operator that maps input signals *x(t)*'s into output signals *y(t)*'s of a given system. Moreover, it is well known that (1) is well determined for all the impulse responses *h(t)*'s as well as signals *x(t)*'s which occur in engineering.

Furthermore, the functions denoted as *h(t)* and *x(t)* in (1), and which occur in engineering, can act as both: system's functions as well as system input signals. To see this, let us introduce an auxiliary variable $t'=t-\tau$ in (1). This leads to

$$y(t)=-\int_{\infty}^{-\infty} h(t-t')x(t')dt'=\int_{-\infty}^{\infty} x(t')h(t-t')dt'\,. \tag{2}$$

And finally, naming the auxiliary variable *t'* by τ, we obtain from (2) the following:

$$y(t)=\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau\,. \tag{3}$$

Note that now in (3) *x(t)* plays a role of a system's function, while *h(t)* a role of an input signal of this system. That is the opposite of what was before, in (1). Or, in other words, this shows we are not able to say what is a system's function and what its input signal in a pair: *h(t)*, *x(t)* – knowing only the output signal *y(t)*. That is *h(t)* and *x(t)* commute with each other (in their roles) in the convolution integral operator given by (1).

The so-called Dirac delta is an object that is also used in the systems theory and signal processing (more generally, in engineering). And, in the engineering literature, it is most often denoted by a symbol $\delta(t)$, what suggests that it could be treated as a function (although we know very well that it is not). However, due to this belief (that it can be handled as a function), it is used by engineers in both roles in (1). That is as a system's function as well as an input signal – in the same way as *h(t)* and *x(t)* considered above.

Let us take now a closer look at this issue. And, consider first the case when *h(t)* in (1) is assumed to be a Dirac impulse. So this allows us to rewrite (1) in the following form:

$$y(t)=\int_{-\infty}^{\infty}\delta(\tau)x(t-\tau)d\tau=x(t). \tag{4}$$

Further, note that the outcome on the right-hand side of equality (4) results from applying the so-called sifting property of the Dirac delta therein. And this allows us to conclude that the Dirac delta makes an identity operator from a convolution one.

The notation of the convolution integral containing a Dirac delta, as in (4), requires one explanation more because such an integral does not in fact exist, neither in the Riemann's sense nor in the Lebesgue's sense. Nevertheless, because of the convenience and habit, this notation is used by engineers for denoting something what must be mathematically understood as a distribution. And this terminological convention will be used in what follows.

Let us check whether $\delta(t)$ and *x(t)* in (4) commute with each other. And, to this end, assume that the operation given by (4) also possesses the properties which were exploited in transforming (1) to (3) – with the intermediate step shown in (2). So performing now the same manipulations as those indicated from (1) to (3) above, that is

$$y(t)=-\int_{\infty}^{-\infty}\delta(t-t')x(t')dt'=\int_{-\infty}^{\infty}x(t')\delta(t-t')dt', \tag{5}$$

we get

$$y(t)=\int_{-\infty}^{\infty}x(\tau)\delta(t-\tau)d\tau=x(t). \tag{6}$$

Comparison of (4) with (6) allows us to say that really $\delta(t)$ and *x(t)* commute with each other (in their roles of a system's function and a system's input signal). At least mathematically, this seems to be fully true and correct.

In the next section, we examine whether physical systems exhibit this property.

2 DIRAC DELTA AND INPUT SIGNAL DO NOT ALWAYS COMMUTE IN DESCRIPTIONS OF LINEAR PHYSICAL SYSTEMS

The flagship example given in engineering textbooks to justify the validity, or even the necessity, of using the Dirac delta concept in describing various physical phenomena is the process of measuring temperature, voltage, current, for example. The author of this paper analyzed critically the process of measuring any physical quantity in general as well as measuring temperature in particular – in the following papers: (Borys A. 2020a), (Borys A. 2020b), (Borys A. 2020c) – in terms of the validity of using Dirac deltas in their descriptions. The results achieved there will serve as a starting point for the considerations of this section.

It has been shown in (Borys A. 2020c) that the process of measuring of a physical quantity (such as, for example, temperature) in time can be described in a similar way as one describes the sampling of a signal of a continuous time. This is shown schematically in Fig. 1.

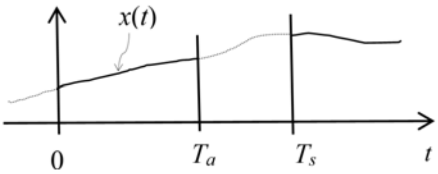

Figure 1. Illustration to modelling a measuring process via the description of sampling signals of a continuous time as discussed in (Borys A. 2020c). In this approach, we assume that the measuring device "delivers" values of the measured physical quantity in a stepwise manner. The length of each step is assumed to be equal to T_s seconds. Further, during each of these steps, it is assumed that the measuring device averages the measured quantity a time equal to $T_a<T_s$. The averaged values are assigned successively to time intervals starting from the beginning of a given step (lasting T_s seconds) to its end (the same value of the measured physical quantity applies to all time instants belonging to a given interval). (This figure is based on a one, which was used in discussions presented in (Borys A. 2020c)).

Denote now by *x(t)* a waveform according to which a physical quantity changes in time (for example, the temperature mentioned above). Obviously, because of the reasons mentioned above, the measuring device is unable to provide us with this waveform in an undistorted manner. Here, we model its behavior as illustrated in Fig. 1 and as described in the caption to this figure. And we look for an analytical expression describing a signal registered by our measuring device; we denote it by *y(t)*.

Let us start with calculation of the value of the signal *y(t)* that is applicable in the interval $0\le t<T_s$. Denote it by $y_a(0\cdot T_s+T_a)=y_a(T_a)$ it will be given by

$$y_a(T_a)=\int_{0}^{T_a}x(t)\varphi(t)dt, \tag{7}$$

where $\varphi(t)$ means an averaging function.

In the next step, to illustrate the averaging operation in time given by (7), let us choose the simplest possible form of $\varphi(t)$ therein that fulfils the conditions for such functions formulated in (Strichartz R. 1994). That function has the following form:

$$\varphi(t)=\begin{cases}1/T_a & \text{for } 0<t\le T_a\\ 0 & \text{elsewhere}\end{cases} \quad . \tag{8}$$

Substituting (8) into (7) gives

$$y_a(T_a)=\frac{1}{T_a}\int_0^{T_a}x(t)\,dt\,. \tag{9}$$

Now we will show that as the following property: $\varphi(t)=\varphi(-(t-T_a))$ holds in the case of the function (8), (7) can be expressed equivalently as a convolution integral. To this end, we rewrite (7) in the following way:

$$\begin{aligned}y_a(T_a)&=\int_0^{T_a}x(t)\varphi(-(t-T_a))\,dt=-\int_{T_a}^{0}x(T_a-\tau)\varphi(\tau)\,d\tau=\\&=\int_0^{T_a}\varphi(\tau)x(T_a-\tau)\,d\tau=\int_{-\infty}^{\infty}\varphi(\tau)x(T_a-\tau)\,d\tau\quad .\end{aligned} \tag{10}$$

In derivation of the final result in (10), we have additionally used an auxiliary variable $\tau=-(t-T_a)$ and the fact that the function $\varphi(t)$ given by (8) is identically equal to zero outside the interval $(0,T_a\rangle$.

In the next step, note that a similar relation as (9) for $y_a(T_a)$ can be written for every $y_a(kT_s+T_a)$, where $k=\ldots,-1,0,1,\ldots$. That is the following one:

$$y_a(kT_s+T_a)=\int_{kT_s}^{kT_s+T_a}x(t)\varphi(t-kT_s)\,dt\,. \tag{11}$$

Further, to get a similar expression as (10), we use the fact that $\varphi(t-kT_s)=\varphi(-(t-kT_s-T_a))$ holds for the function $\varphi(t)$ given by (8). So (11) can be re-written as

$$\begin{aligned}y_a(kT_s+T_a)&=\int_{kT_s}^{kT_s+T_a}x(t)\varphi(-(t-kT_s-T_a))\,dt=\\&=-\int_{T_a}^{0}x(kT_s+T_a-\tau)\varphi(\tau)\,d\tau=\int_0^{T_a}\varphi(\tau)x(kT_s+T_a-\tau)\,d\tau=\\&=\int_{-\infty}^{\infty}\varphi(\tau)x(kT_s+T_a-\tau)\,d\tau\quad .\end{aligned} \tag{12}$$

Note that in derivation of the final result in (12), we have used an auxiliary variable $\tau=-(t-kT_s-T_a)$ and, similarly as before, the fact that the function $\varphi(t)$ given by (8) is identically equal to zero outside the interval $(0,T_a\rangle$.

Having derived the results (12) and (10) (where the latter is a special case of (12) for $k=0$), we are able now to express the signal $y(t)$ for all times. It will be given by

$$y(t)=y_a(kT_s+T_a)=\int_{-\infty}^{\infty}\varphi(\tau)x(kT_s+T_a-\tau)\,d\tau \tag{13}$$

for t belonging to the successive time intervals $kT_s\le t<(k+1)T_s$ when k assumes successively the values $k=\ldots,-1,0,1,\ldots$.

Further, observe that the function given by (13) is a step function with the values of its steps equal to the corresponding $y_a(kT_s+T_a)$'s, $k=\ldots,-1,0,1,\ldots$ occurring in the successive time segments $kT_s\le t<(k+1)T_s$, $k=\ldots,-1,0,1,\ldots$.

It is interesting to note that the function given by (13) can be expressed also in another way, as a sum of some functions. And, to see this, let us start with defining first these functions; we define them in the following way:

$$y_{kT_s}(t)=\begin{cases}y_a(kT_s+T_a)=\int_{-\infty}^{\infty}\varphi(\tau)x(kT_s+T_a-\tau)\,d\tau\\ \text{for } kT_s\le t<(k+1)T_s\\ \text{and}\\ 0 \quad \text{outside the above range of } t\text{'s}\end{cases} \tag{14}$$

with k's in (14) that may take the following values: $\ldots,-1,0,1,\ldots$. So, with the help of the functions given by (14), we can express $y(t)$ from (13) in a compact way as follows:

$$y(t)=\sum_{k=-\infty}^{\infty}y_{kT_s}(t)\,. \tag{15}$$

As already said in Introduction, in various technical disciplines which use descriptions in form of convolution integrals, it is assumed that what stands on the left-hand side under a convolution integral is related with some operator (operation) performed on a signal (physical quantity) varying in time – the latter standing on the right-hand side under the above integral. Obviously, this matter of occupied position is a matter of convention, but it has its justification in what the convolution integral is used for in engineering sciences. Figuratively speaking, we could express this in the following way: a convolution integral weaves together two roles: of a transforming operation (performed by a system considered) and of being a signal (physical quantity), which is subjected to the action of the former. And as already said, the first role is customarily assigned to the left-hand side under the integral, and the second to its right-hand side.

So, now with regard to the convolution integrals occurring in the expressions (13) and (14), the function $\varphi(t)$ is playing therein a role of a transforming operation, but the function $x(t)$ a role of a physical quantity (for example, of a temperature varying with time). And, as already known from the considerations presented in Introduction, as long as these functions remain "decent" (what we mean under this term is explained below), they can perform both roles. That is they can stand in a convolution integral on both positions: being the left-hand side as well as the right-hand side of the expression under the integral. Unfortunately, when considering concrete physical systems this is not always the case. In what follows, we explain this point on an example of measuring a time-varying temperature; this example is considered throughout the paper.

As we know, temperature as a physical quantity is bounded. For example, let us consider the temperature on Earth. We can say that this temperature does not exceed the lower limit of -100 degrees Celsius and the upper limit of +100 degrees Celsius. Consider it as changing with the passage of time: opassing hours, days, years. So it will be represented by a function of time. Further, let us identify it with the function *x(t)* introduced previously. So it will be a bounded function for which we can write

$$\left|x(t)\right| < M \text{ for every } t, \tag{16}$$

where *M* denotes the bounding constraint imposed on the function *x(t)*.

Let us now take such a $\varphi(t)$ function occurring in (13) and (14) which does not exhibit the constraint given in (16). That is there are possible absolute values of $\varphi(t)$ which exceed the value of *M*. In this case, obviously, the functions *x(t)* and $\varphi(t)$ cannot change their roles in (13) and (14) because $\varphi(t)$ so chosen is not a physically reasonable function that describes the temperature changes on Earth. In other words, the above functions *x(t)* and $\varphi(t)$ do not commute (their roles do not commute) in the integrals in (13) and (14) because of physical reasons.

Of course, by dropping the condition (16) for the function *x(t)*, we "restore" the commutativity property of the functions *x(t)* and $\varphi(t)$ in the integrals in (13) and (14), but at the cost that the function *x(t)* will not be able to be interpreted as a function that determines temperature changes on Earth.

As we will see further on, the lack of commutativity property of certain functions $\varphi(t)$ with the function describing temperature changes on Earth will manifest itself in full as we move in the formulas (13) and (14) from a finite-time averaging operation (i.e. with a finite) to "ideal" averaging in time, i.e. with the value of the parameter T_a going to zero.

The result presented in this section, which indicates possibility of the lack of commutativity property between an input signal at the input of a linear system and its so-called system's function – in a description of that system, may seem a little bit strange. We are accustomed to the fact that the aforementioned property takes place. However, note that the fact that this is not always the case has already been pointed out by others, for example by Irwin Sandberg in the following papers: (Sandberg I. 2008) and (Sandberg I. 2000). So, really, the commutativity property is not obligatory in linear systems.

3 IDEAL AVERAGING

Let us now consider the case of a temperature measurement, as in the example of the previous section, where the averaging operation is performed at ever shorter time intervals. Note that such a scenario is referred to in the literature, for example in (Strichartz R. 1994) to justify the need for the use of a Dirac delta. So, now, we will assume that in our averaging function $\varphi(t)$, given by (8), the parameter T_a goes to zero. Thus, this function will approach the Dirac's delta – in the sense of the series-based distribution theory (see, for example, (Hoskins R. F. 2010), (Strichartz R. 1994) – in the integrals in the expressions (13) and (14). And these formulas will take then the following forms:

$$y_i(t) = y_{ai}(kT_s) = \int_{-\infty}^{\infty} \delta(\tau)\, x(kT_s - \tau)\, d\tau = x(kT_s) \tag{17}$$

for t belonging to the successive time intervals $kT_s \le t < (k+1)T_s$ when k assumes successively the values *k*=*…,-1,0,1,…,* and

$$y_{kT_s i}(t) = \begin{cases} y_{ai}(kT_s) = \int_{-\infty}^{\infty} \delta(\tau)\, x(kT_s - \tau)\, d\tau = x(kT_s) \\ \text{for } kT_s \le t < (k+1)T_s \\ \text{and} \\ 0 \quad \text{outside the above range of } t\text{'s} . \end{cases} \tag{18}$$

In (17) and (18), the values of $y_{ai}(kT_s)'s$ stand for the corresponding $y_a(kT_s)'s$ calculated in the case of considering an ideal averaging; that is with the one in which the parameter $T_a \to 0$. Obviously, the latter means that the system's function in this case $\varphi(t) \to \delta(t)$ (in the sense explained, for example, in (Hoskins R. F. 2010) and (Strichartz R. 1994)). And just because of this reason, we speak here about an ideal averaging (extending the subscript *a* at $y_a(kT_s)'s$ to *ai*). Moreover, for the same reasons, the letter "*i*" is also added to as an subscript at *y(t)* in (17) and for extending subscripts at $y_{kT_s}(t)$'s in (18), i.e. to visualize $y(t) \to y_i(t)$ and *k*=*…,-1,0,1,…*.

Taking into account the above changes in indices requires (15) to be rewritten, too; namely as

$$y_i(t) = \sum_{k=-\infty}^{\infty} y_{kT_s i}(t) . \tag{19}$$

Furthermore, note that the function $y_i(t)$ given by (17) or (19) remains a step function (as its "non-ideal" version given by (13) or (15)). Its steps in the successive time intervals: $kT_s \le t < (k+1)T_s$, , *k*=*…,-1,0,1,…,* will be equal to the values of the function *x(t)* at the successive time instants kT_s, *k*=*…,-1,0,1,…*.

We draw also the reader's attention to the fact that the function $y_i(t)$ due to its shape as a step function is not identical with the function *x(t)*. In other words, the following:

$$y_i(t) \ne x(t) \tag{20}$$

holds.

Finally in this section, note that the function *x(t)* cannot replace in any way that action of the Dirac delta (i.e. the action of performing an ideal averaging), which we see in (17) or (18). Simply because of the constraint (16) imposed on this function, which makes it impossible to assume that it can grow to infinity for some times – as it was possible with the function $\varphi(t) \to \delta(t)$ in (13) and (14) (in the sense of the series-based distribution theory (see, for example, (Hoskins R. F. 2010), (Strichartz R. 1994))). And see that this further reinforces what we discovered in the previous section. Namely that the roles of the functions $\varphi(t)$ and $x(t)$ in description of the temperature measurement with the use of the relation (13) or (15) do not commutate with each other. In general, the function $x(t)$ should not be interpreted in this case as a system's function and $\varphi(t)$ as a signal at the input of the measurement device (system). Always the opposite should occur. That is the function $x(t)$ should be identified with the signal applied to a system's input, but $\varphi(t)$ should be identified with the system's function of this system.

4 DIRAC DELTA AS A TECHNICAL MEANS TO DETERMINE THE IMPULSE RESPONSE OF A LINEAR SYSTEM

A commonly used method in the literature (Vlach J., Singhal K. 1983), (Sandberg I. 2003) for determining the system's function (called also the impulse response of a system) of systems having descriptions in form of the convolution integral is to apply a Dirac delta that is assumed then to be an input signal. However, as well known, such signals are not really encountered in engineering. So the Dirac impulse should be then treated more as a technical means for calculating a system's function rather than a real signal. Formally, see that applying $x(t) = \delta(t)$ in (1), we get

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\delta(t-\tau)\,d\tau = h(t). \tag{21}$$

That is then the output signal of a system is equal to its system's function $h(t)$, which, just because of the application of the Dirac impulse at the input of a system, is called its impulse response.

In the context of the above, note that to get the result given by (21), we assumed in fact, tacitly, that in the natural description of a linear system by a convolution integral the unbounded input signals are admissible therein. And just this assumption allowed us to use $x(t)=\delta(t)$ in (1) to get (21).

But what to do when the input signals in the convolution integral description (1) are not allowable to exceed some values? As, for instance, in the example analyzed in the previous section (see the constraint (16)). Is it possible to determine the system's function from (1) despite the above restriction or rather not?

We can reason in this case as follows. Let us insert into (1) successive functions coming from a sequence approximating the Dirac impulse in the sense of the series-based distribution theory (see, for example, (Hoskins R. F. 2010), (Strichartz R. 1994)) and check each time whether the calculated convolution integral exists. And finally, check whether this infinite sequence of integrals possesses its limit for every time instant (i.e. a limit function). If yes, one must conclude that this procedure makes sense. And, we get a useful result that provides us with the system's function, according to (21).

Note, however, that the procedure described above is only partially applicable in (13) and (14): only at those places, where the calculation of the values of $\varphi(kT_s + T_a)$ is performed. In more detail, we get then from (13) and (14)

$$y_\delta(t) = \varphi(kT_s + T_a) = \int_{-\infty}^{\infty} \varphi(\tau)\delta(kT_s + T_a - \tau)\,d\tau \tag{22}$$

for t belonging to the successive time intervals $kT_s \le t < (k+1)T_s$ when k assumes successively the values $k=\ldots,-1,0,1,\ldots$, and

$$y_{kT_s\delta}(t) = \begin{cases} \varphi(kT_s + T_a) = \int_{-\infty}^{\infty} \varphi(\tau)\delta(kT_s + T_a - \tau)\,d\tau \\ \text{for } kT_s \le t < (k+1)T_s \\ \text{and} \\ 0 \quad \text{outside the above range of } t\text{'s}, \end{cases} \tag{23}$$

respectively. The indices k's in (23) may take the following values: $\ldots,-1,0,1,\ldots$.

Further, from (15), we obtain

$$y_\delta(t) = \sum_{k=-\infty}^{\infty} y_{kT_s\delta}(t) \tag{24}$$

in the case considered. Moreover, note that the function $y_\delta(t)$ in (22) or in (24) means $y(t)$ in (13) or in (15) for a particular $x(t)=\delta(t)$. Similarly, $y_{kT_s\delta}(t)$'s in (23) stand for $y_{kT_s}(t)$'s in (14) for a particular $x(t)=\delta(t)$.

In the next step, see that the function $y_\delta(t)$ calculated in (22) or in (24), when the function $\varphi(t)$ is given by (8), assumes the following form:

$$y_\delta(t) = \begin{cases} \varphi(T_a) = \dfrac{1}{T_a} \\ \text{for } 0 \le t < T_s \\ \text{and} \\ 0 \quad \text{outside the above range of } t\text{'s}. \end{cases} \tag{25}$$

Comparison of the function $y_\delta(t)$ given by (25) with the function $\varphi(t)$ given by (8) shows that they differ from each other.

One may ask why this happens. The answer is rather obvious. A mapping of the signal $x(t)$ to the measured one, $y(t)$, performed by a measuring equipment consists not only of a locally performed convolution operations (convolution integrals). It also includes a momentary (delayed) holding of the "worked out" average value in the measuring device.

5 CONCLUSIONS

Under assumption of an ideal operation of some systems, the so-called Dirac deltas (Dirac impulses) appear in their descriptions. Unfortunately, in many textbooks and papers, they become a source of misinterpretations and errors. One of such basic errors lies in the fact that the Dirac impulse is uncritically assumed to be a one of the possible signals that can appear in the ideal description of a given system. That is it can be treated interchangeably with its so-called impulse response. But it cannot, and this is pointed out in this paper. An example of a device, which measures temperature, was used here to illustrate the analysis, derivations and discussion presented. Another example of this type, coming from the theory of sampling ideally analog signals, is discussed in another work (Borys A. 2023) of the author of this paper.

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