TRANSNAV
http://www.transnav.eu
the International Journal on Marine Navigation and Safety of Sea Transportation
Volume 13
Number 3
September 2019

DOI: 10.12716/1001.13.03.09

Derivation of Formulas in Spherical Trigonometry Based on Rotation Matrix

T.H. Hsieh, S.Z. Wang, W. Liu & J.S. Zhao
Shanghai Maritime University, Shanghai, China
C.L. Chen
National Taiwan Ocean University, Keelung, Taiwan

ABSTRACT: The formulas of spherical triangle, which are widely used to solve various navigation problems, are the important basic knowledge of nautical mathematics. Because the sine rules and the cosine rules for the sides are the fundamental formulas to derive the other spherical triangle formulas, they are also called the genetic codes of the spherical triangle formulas. In the teaching process, teachers usually use the geometric method to derive and prove these fundamental formulas. However, the derivation of geometric methods is complicated and difficult to understand. To improve the teaching process, this paper proposes the three-dimensional rotation method, which is based on conversion of two cartesian coordinate frames using the rotation matrices. This method can easily and simultaneously derive the sine rules, the cosine rules for the sides, and the five-part formulas (I), and is also helpful to solve different kinds of spherical navigation problems.

1 INTRODUCTION

The formulas of spherical triangle, which include many different kinds of formulas such as the sine rules, the cosine rules for the sides, the four-part formulas, the five-part formulas, etc. are the important basic knowledge of nautical mathematics. They are widely applied to solve the various navigation problems. Hsu et al. have provided an architecture diagram to clarify the relationships of these formulas (Hsu et al., 2005). On this basic, we add the formulas of the right spherical triangle and the formulas of the quadrantal spherical triangle to reorganize the diagram as shown in Figure 1. For instance, the sine rules can be derived from the cosine rules for the sides. The four-part formulas and the five-part formulas (I) can be derived from the sine rules and the cosine rules for the sides. The five-part formulas (II) can be derived from the cosine rules for the sides or the polar duality property of five-part formulas (I), etc. In addition, the two special cases, the right spherical triangle which one angle is 90°and the quadrantal spherical triangle which one side is 90°, can be derived from the cosine rules for the sides, the sine rules, and the four-part formulas. As shown in Figure 1, the cosine rules for the sides and the sine rules are the fundamental formulas to derive the other spherical triangle formulas, hence, they are also called the genetic codes of the spherical triangle formulas. To derive these formulas, many scholars have proposed different approaches as follows: Clough-Smith (1966), Todhunter (1886) and Smart (1977) have presented a geometric method to derive the cosine rules for the sides and the sine rules, Green (1985) has provided a vector method to derive the cosine rules for the sides and the sine rules, etc. However, in the teaching process, when the teacher uses the geometric method to derive and prove the sine rules and the cosine rules for the sides, students usually feel the process is complicated and difficult to understand.

Although using the vector method instead may let the process become simpler, different fundamental formulas still need to be derived respectively. Thus, this paper proposes the three-dimensional rotation method to convert between two cartesian coordinate frames using the rotation matrices. This method can easily and simultaneously derive the sine rules, the cosine rules for the sides, and the five-part formulas (I).

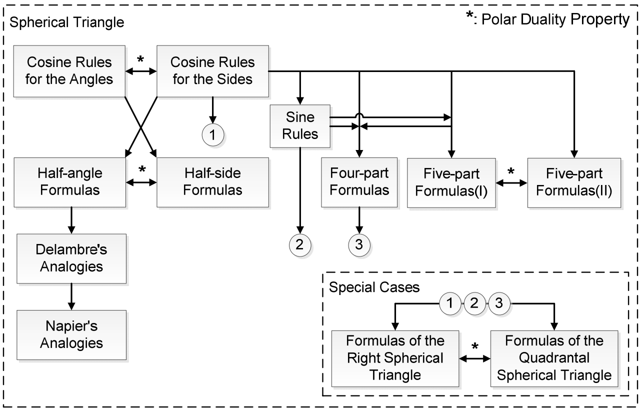

Figure 1. Relationships of the various spherical triangle formulas.

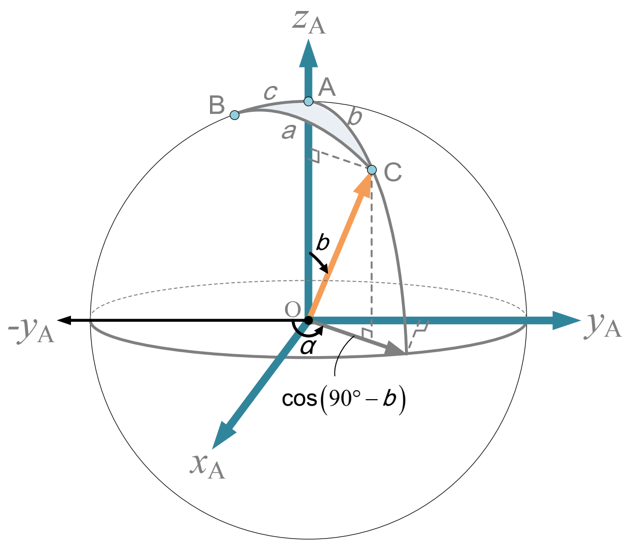

Figure 2. Locating the point C in the frame A.

2 DERIVATION OF SPHERICAL TRIANGLE FORMULAS

2.1 *Two cartesian coordinate frames are defined*

A spherical triangle ($\widehat{\Delta ABC}$) on an unit sphere has six elements which include the three sides (a, b, and c) and three angles (α, β, and γ). First, we set up the frame A (x_A, y_A, z_A) and select the point A as a pole of the frame A on an unit sphere, as shown in Figure 2. The coordinates of the point C on the unit sphere in the frame A (C_A) is as follows:

$$C_A = \begin{bmatrix} \cos(90° - b)\sin(180° - \alpha) \\ \cos(90° - b)\cos(180° - \alpha) \\ \sin(90° - b) \end{bmatrix} \quad (1)$$

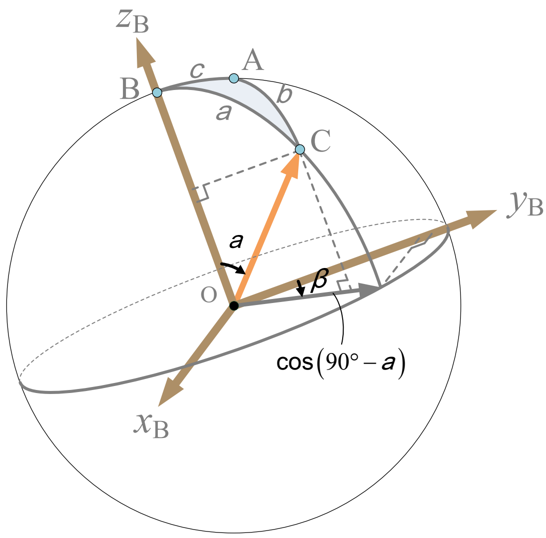

Figure 3. Locating the point C in the frame B.

Then, we create the frame B (x_B, y_B, z_B) and set the axis Z of the frame B intersects an unit sphere at the point B, as shown in Figure 3. The coordinates of the point C on the unit sphere in the frame B (C_B) is as follows:

$$C_B = \begin{bmatrix} \cos(90° - a)\sin\beta \\ \cos(90° - a)\cos\beta \\ \sin(90° - a) \end{bmatrix} \quad (2)$$

2.2 *Transformation of coordinate frames on basic rotation*

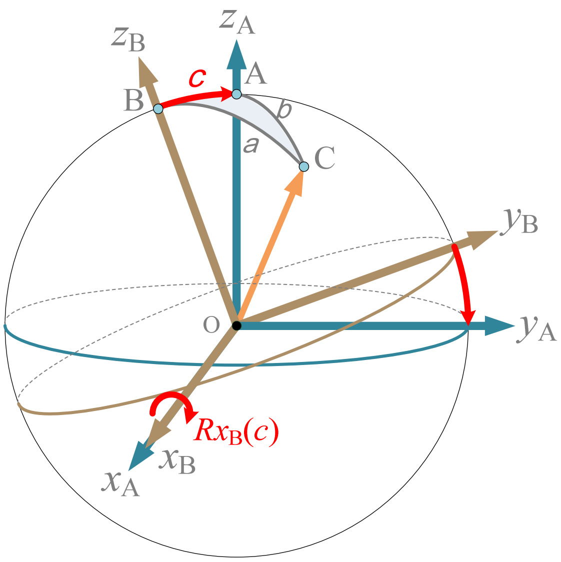

Figure 4. Conversion of frame B to frame A using the clockwise rotation matrix.

As shown in Figure 4, when the frame A and the frame B are displayed on the same figure and the origin is the same, the coordinates of the point C in both frames (C_A and C_B) represent the position of the same point C. When the frame B is rotated clockwise the angle of the side c along the axis x_B to overlap it with the frame A, the coordinates C_B can also be converted to coordinates C_A by using the clockwise rotation matrix as shown in Equation 3. The clockwise rotation matrix of the axis x is listed in Equation 4 (Arfken, 1985).

$$C_A = Rx_B(c)C_B \tag{3}$$

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \tag{4}$$

Substitute Equations 1, 2 and 4 into Equation 3, and rearrange the formulas as follows:

$$\sin b \sin\alpha = \sin a \sin\beta \tag{5}$$

$$-\sin b \cos\alpha = \cos c \sin a \cos\beta - \sin c \cos a \tag{6}$$

$$\cos b = \sin c \sin a \cos\beta + \cos c \cos a \tag{7}$$

And then, as shown in Figure 5, when the frame A is rotated counterclockwise the angle of the side c along the axis x_A to overlap it with the frame B, the coordinates C_A can also be converted to coordinates C_B by using the counterclockwise rotation matrix as shown in Equation 8. The counterclockwise rotation matrix of the axis x is listed in Equation 9 (Arfken, 1985).

$$C_B = R'x_B(c)C_A \tag{8}$$

$$R'_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \tag{9}$$

Substitute Equations 1, 2 and 9 into Equation 8, and rearrange the formulas as follows:

$$\sin a \sin\beta = \sin b \sin\alpha \tag{10}$$

$$\sin a \cos\beta = -\cos c \sin b \cos\alpha + \sin c \cos b \tag{11}$$

$$\cos a = \sin c \sin b \cos\alpha + \cos c \cos b \tag{12}$$

Equations 5 and 10 are the sine rules; Equations 6 and 11 are the five-part formulas (I); Equations 7 and 12 are the cosine rules for the sides. The results show that the three-dimensional rotation method can derive the sine rules, the cosine rules for the sides and the five-part formulas (I) simultaneously.

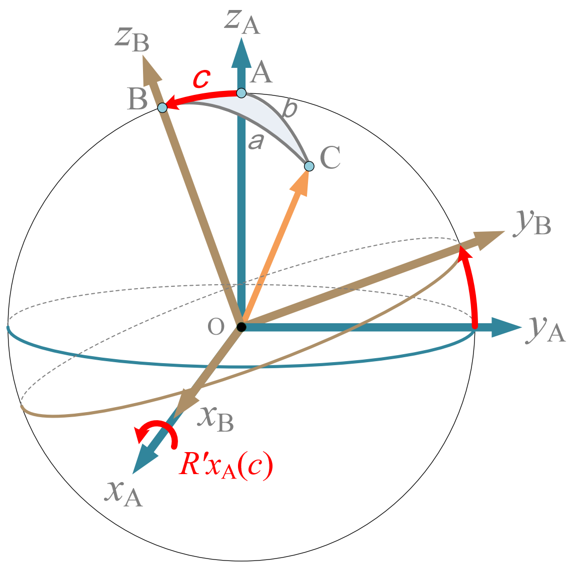

Figure 5. Conversion of frame A to frame B using the counterclockwise rotation matrix.

2.3 *Element substitution*

Other formulas of the sine rules, the cosine rules for the sides and the five-part formulas can be obtained by using the element substitution. In Section 2.2, the position order of elements in the spherical triangle are as shown in Figure 6(a). To replace the elements in Figure 6(a) with the elements in Figure 6(b), Equations 5, 6 and 7 can be replaced to Equations 13, 14 and 15; Equations 10, 11 and 12 can be replaced to Equations 16, 17 and 18 respectively.

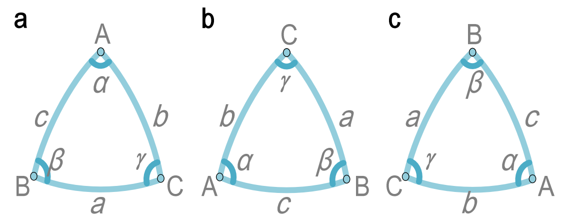

Figure 6. Position order of elements.

$$\sin a \sin \gamma = \sin c \sin \alpha \tag{13}$$

$$-\sin a \cos \gamma = \cos b \sin c \cos \alpha - \sin b \cos c \tag{14}$$

$$\cos a = \sin b \sin c \cos \alpha + \cos b \cos c \tag{15}$$

$$\sin c \sin \alpha = \sin a \sin \gamma \tag{16}$$

$$\sin c \cos \alpha = -\cos b \sin a \cos \gamma + \sin b \cos a \tag{17}$$

$$\cos c = \sin b \sin a \cos \gamma + \cos b \cos a \tag{18}$$

Equations 13 and 16 are the sine rules; Equations 14 and 17 are the five-part formulas (I); Equations 15 and 18 are the cosine rules for the sides.

Similarly, to replace the elements in Figure 6(a) with the elements in Figure 6(c), Equations 5, 6 and 7 can be replaced to Equations 19, 20 and 21; Equations 10, 11 and 12 can be replaced to Equations 22, 23 and 24 respectively.

$$\sin c \sin \beta = \sin b \sin \gamma \tag{19}$$

$$-\sin c \cos \beta = \cos a \sin b \cos \gamma - \sin a \cos b \tag{20}$$

$$\cos c = \sin a \sin b \cos \gamma + \cos a \cos b \tag{21}$$

$$\sin b \sin \gamma = \sin c \sin \beta \tag{22}$$

$$\sin b \cos \gamma = -\cos a \sin c \cos \beta + \sin a \cos c \tag{23}$$

$$\cos b = \sin a \sin c \cos \beta + \cos a \cos c \tag{24}$$

Equations 19 and 22 are the sine rules; Equations 20 and 23 are the five-part formulas (I); Equations 21 and 24 are the cosine rules for the sides.

2.4 *Derivation of other spherical triangle formulas*

The other spherical triangle formulas can be easily derived from the sine rules, the cosine rules for the sides, and the five-part formulas (I). For example, the four-part formulas can be derived from the sine rules and the five-part formulas (I) as follows: Substituting Equation 5 into 6 can yield Equation 25; substituting Equation 10 into 11 can yield Equation 26; substituting Equation 13 into 14 can yield Equation 27; substituting Equation 16 into 17 can yield Equation 28; substituting Equation 19 into 20 can yield Equation 29; substituting Equation 22 into 23 can yield Equation 30.

$$\cos \beta \cos c = \cot a \sin c - \cot \alpha \sin \beta \tag{25}$$

$$\cos \alpha \cos c = \cot b \sin c - \cot \beta \sin \alpha \tag{26}$$

$$\cos \alpha \cos b = \cot c \sin b - \cot \gamma \sin \alpha \tag{27}$$

$$\cos \gamma \cos b = \cot a \sin b - \cot \alpha \sin \gamma \tag{28}$$

$$\cos \gamma \cos a = \cot b \sin a - \cot \beta \sin \gamma \tag{29}$$

$$\cos \beta \cos a = \cot c \sin a - \cot \gamma \sin \beta \tag{30}$$

Equations 25 to 30 are the complete four-part formulas.

3 METHODS COMPARISON

When we use the geometric method to derive the fundamental formulas, the process is complicated and different. As shown in Figure 7, we have to draw an auxiliary plane triangle ΔADE outside the unit sphere, and then use the plane cosine rules and Pythagorean equations to derive the cosine rules for the sides of the sphere triangle ΔABC. However, if we want to use the geometric method to derive the sine rules, the derivation process is completely different. We must draw two auxiliary plane triangles inside the unit sphere, as shown in Figure 8.

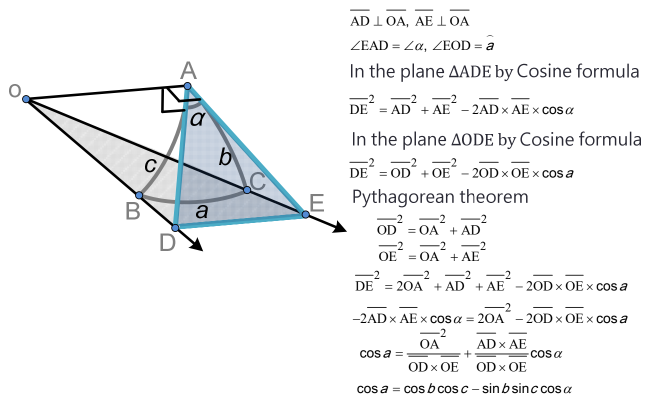

Figure 7. Using geometric method to derive the cosine rules for the sides.

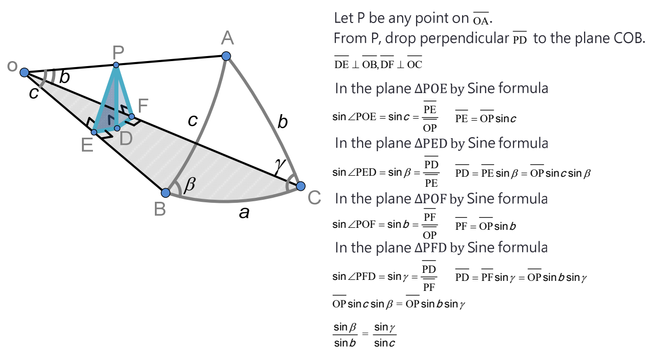

Figure 8. Using geometric method to derive the sine rules.

In addition, as shown in Figure 9 and Figure 10, using the vector method to derive the fundamental formulas may let the process become simpler, but the process is also different. In comparison, the three-dimensional rotation method proposed in this paper can easily and simultaneously derive the sine rules, the cosine rules for the sides, and the five-part formulas (I). It is especially helpful to the teaching process.

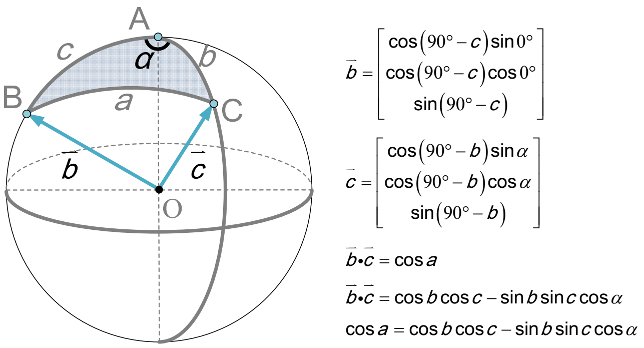

Figure 9. Using vector method to derive the cosine rules for the sides.

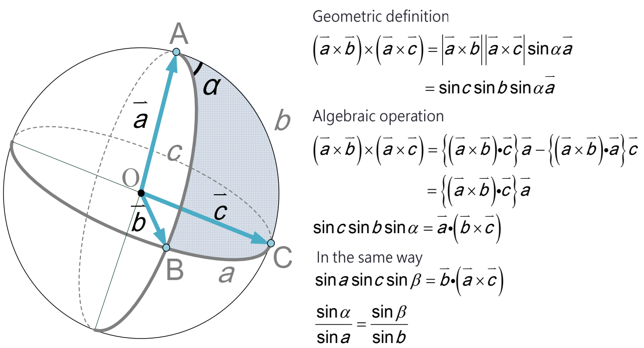

Figure 10. Using vector method to derive the sine rules.

4 APPLICATION OF SPHERICAL TRIANGLE FORMULAS

4.1 *Problem of great circle track*

The problems involving the relationships between the three sides and three angles of the spherical triangle can be solved by using the spherical triangle formulas. The problem of great circle track is one of them. As shown in Figure 11, while adopting the great circle track form the point F to the point T, the latitude and the longitude of the two points are given to find the great circle distance (D_{FT}) and the initial course angle (C).

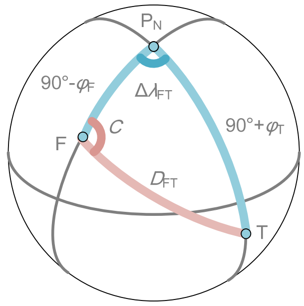

Figure 11. Problem of great circle track.

In this problem, the formula of the great circle distance can be obtained by using the cosine rules for the sides as shown in Equation 31, and the formula of the initial course angle can be obtained by using the four-part formulas as shown in Equation 32.

$$\cos D_{\mathrm{FT}} = \sin\varphi_{\mathrm{F}}\sin\varphi_{\mathrm{T}} + \cos\varphi_{\mathrm{F}}\cos\varphi_{\mathrm{T}}\cos\Delta\lambda_{\mathrm{FT}} \quad (31)$$

$$\tan C = \frac{\sin\Delta\lambda_{\mathrm{FT}}}{\cos\varphi_{\mathrm{F}}\tan\varphi_{\mathrm{T}} - \sin\varphi_{\mathrm{F}}\cos\Delta\lambda_{\mathrm{FT}}} \quad (32)$$

4.2 *Demonstrated example*

A ship leaves from New York (φ 40°27.1′ N, λ 73°49.4′ W) to Cape Town (φ 33°53.3′ S, λ 18°23.1′ E). The latitude of the departure point ($\varphi_{\mathrm{F}} = 40.452°$), the difference of longitude ($\Delta\lambda_{\mathrm{FT}} = 92.208°$), and the latitude of the destination point ($\varphi_{\mathrm{T}} = -33.888°$) are given. The captain wants to know the great circle distance and the initial course (Bowditch, 1981). By using Equation 31, we can obtain the great circle distance which is 6762.7 nautical miles ($D_{\mathrm{FT}} = 112.712°$). By using Equation 32, we can yield the initial course which is 115.9° ($C = \mathrm{N}115.942°\mathrm{E}$).

5 CONCLUSIONS

In this paper, we provide the architecture diagram to clarify the relationships of various spherical triangle formulas and indicate that the sine rules and the cosine rules for the sides are the fundamental formulas to derive the other spherical triangle formulas. In addition, we propose the three-dimensional rotation method which can derive the sine rules, the cosine rules for the sides, and the five-part formulas (I) simultaneously. The method is easier to understand and can improve the teaching process. Furthermore, in practical use, we also demonstrate how to use the spherical triangle formulas to solve the spherical navigation problems, such as finding the great circle distance and the initial course angle.

REFERENCES

Hsu T. P., Chen C. L., Chang J. R. (2005). New Computational Methods for Solving Problems of the Astronomical Vessel Position. *The Journal of Navigation*, 58(2): 315-335.

Clough-Smith J. H. (1966). An Introduction to Spherical Trigonometry. Brown, Son & Ferguson, Ltd.

Todhunter I. (1886). Spherical Trigonometry: For the Use of Colleges and Schools", Fifth ed., Macmillan and Company.

Smart W. M. (1977). Text-Book on Spherical Astronomy, Sixth ed., Cambridge University Press.

Green R. M. (1985). Spherical Astronomy, Cambridge University Press.

Arfken G. (1985). Mathematical Methods for Physicists, Third ed., Academic Press.

Bowditch N. (1981). American Practical Navigator, Pub. 9, Vol. 2, Defense Mapping Agency Hydrographic/Topographic Center.