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Definition of Sampled Signal Spectrum and Shannon's Proof of Reconstruction Formula

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ABSTRACT: The objective of this paper is to show from another perspective that the definition of the spectrum of a sampled signal, which is used at present by researchers and engineers, is nothing else than an arbitrary choice for what is possibly not uniquely definable. To this end and for illustration, the Shannon's proof of reconstruction formula is used. As we know, an auxiliary mathematical entity is constructed in this proof by performing periodization of the spectrum of an analog, bandlimited, energy signal. Admittedly, this entity is not called there a spectrum of the sampled signal - there is simply no need for this in the proof – but as such it is used in signal processing. And, it is not clear why just this auxiliary mathematical object has been chosen in signal processing to play a role of a definition of the spectrum of a sampled signal. We show here what are the interpretation inconsistences associated with the above choice. Finally, we propose another, simpler and more useful definition of the spectrum of a sampled signal, for the cases where it can be needed.

1 INTRODUCTION

The signal spectrum is its image or, in other words, its representation in the frequency domain. And, it is generally agreed among researchers and engineers that all the non-pathological, deterministic, and non-periodic signals exploited in the area of signal processing possess such the representation. It is a Fourier transform of a given signal.

Similarly, all the periodic signals can be represented by their Fourier series. So, their frequency images can be considered as discrete spectra, what is visualized in Fig. 1.

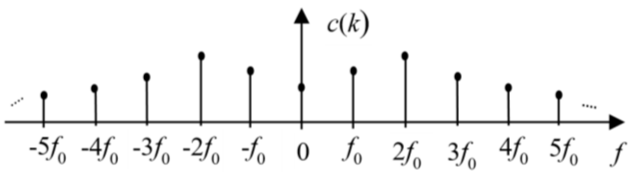

Figure 1. Visualization of the discrete spectrum of an example periodic signal.

In Fig. 1, $c(k)$ means the *k*-th coefficient in the Fourier series of a periodic signal considered.

Note that in general the coefficient $c(k)$ is a complex number. It is connected with the frequency kf_0, what makes possible to treat the set of all $c(k)$'s as a set of values of a certain function of f (spectrum). Because of this reason $c(k)$ is used here to denote also, in short, this function. Moreover, f and f_0 in Fig. 1 stand for the frequency variable and the so-called fundamental frequency (fundamental harmonic) in the spectrum of this signal.

The common belief among researchers and engineers is that both kinds of the spectra mentioned above, i.e. the continuous spectrum related with the Fourier transform and the discrete one connected with the Fourier series, fit to each other in some way. However, this is only an illusion. Why? Because of many reasons. But, probably, the most important one follows from the fact that the spectrum of a periodic signal, say $x_p(t)$, where t means a continuous time variable, cannot be calculated in a "normal way" via the Fourier transform.

What we understand by the "normal way of calculation" mentioned above is illustrated in (1) below:

$$\begin{aligned} X_p(f) &= \int_{-\infty}^{\infty} x_p(t)\exp(-j2\pi ft)\,dt = \\ &+.....+\int_{-3T/2}^{-T/2} x_p(t)\exp(-j2\pi ft)\,dt + \\ &+\int_{-T/2}^{T/2} x_p(t)\exp(-j2\pi ft)\,dt + ... = \\ &= \infty \cdot X_{PART}(f) = \infty \ . \end{aligned} \tag{1}$$

In (1), $T = 1/f_0$ denotes the period of the periodic signal $x_p(t)$, $X_p(f)$ its Fourier transform (if exists?), $j = \sqrt{-1}$, and the function $X_{PART}(f)$ is one of the identical components of an infinite sum there. Obviously, this component cannot be identically equal to zero for all frequencies. Therefore, there are bands of frequencies for which (1) results in infinite values. And, this is the interpretation of what is expressed in (1).

Further, the above description of $X_p(f)$ shows that by no means it can resemble that what is presented in Fig. 1. So, because of this fact, it seems that only reasonable conclusion here is the following one: signal spectra obtained as their Fourier transforms are not compatible with those following from the Fourier expansions (having the form of the one visualized in Fig. 1). And, obviously, this is a huge problem in cases where both the periodic and non-periodic signals occur together in a system or circuit, in a mixed form. One of the representative examples here are the operations of sampling of an analog signal and its reconstruction from a sequence of its discrete values.

As we know very well, there is a theory (Marks II R. J. 1991), (Vetterli M., Kovacevic J., Goyal V. K. 2014), (Oppenheim A. V., Schafer R. W., Buck J. R. 1998), (Bracewell R. N. 2000), (McClellan J. H., Schafer R., Yoder M. 2015), (Brigola R. 2013), (So H. C. 2019), (Wang R. 2010), (Ingle V. K., Proakis J. G. 2012), (Jenkins W. K. 2009) (to mention only a few of excellent textbooks on fundamentals of digital signal processing), which overcomes the problem sketched above. But, this theory applies non-physical objects called Dirac distributions (named also Dirac generalized functions or Dirac deltas) (Schwartz L. 1950-1951), (Dirac P. A. M. 1947).

With application of this theory, (1) results in a solution, which is a sum of Dirac deltas multiplied by real numbers – these numbers are the corresponding coefficients of the Fourier expansion of $x_p(t)$. So, as a consequence, the image of the discrete spectrum of a periodic signal – as it is visualized in Fig. 1 – must be then modified accordingly. Then, in case of our example signal, it has the form presented in Fig. 2 and is denoted by $X_{p,D}(f)$.

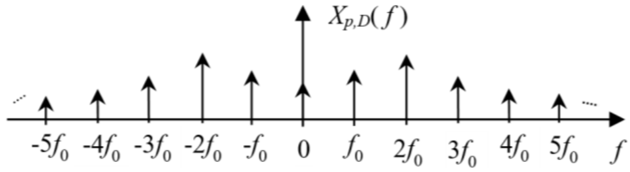

Figure 2. Visualization of the discrete spectrum of an example periodic signal after a model that results in Dirac deltas in (1).

The arrows in Fig. 2 represent the frequency-shifted Dirac deltas $\delta(f - kf_0)$ multiplied by the corresponding coefficients $c(k)$ of the Fourier series of the signal $x_p(t)$. So, the spectra presented in Fig. 1 and in Fig. 2 are not identical; they are two different images of the signal $x_p(t)$ in the frequency domain. But, this is allowed in the theory that is currently in force.

Note however that the above philosophy allowing a signal to have two (or more) different spectra can be a source of confusions and misinterpretations. One notable example of this kind is a strong belief of researchers and engineers that there occur aliasing and folding effects in spectra of sampled signals (sampled in an ideal way) (Marks II R. J. 1991), (Vetterli M., Kovacevic J., Goyal V. K. 2014), (Oppenheim A. V., Schafer R. W., Buck J. R. 1998), (Bracewell R. N. 2000), (McClellan J. H., Schafer R., Yoder M. 2015), (Brigola R. 2013), (So H. C. 2019), (Wang R. 2010), (Ingle V. K., Proakis J. G. 2012), (Jenkins W. K. 2009). Note that their mistake in this case lies in the fact that they draw their conclusions from the analysis of the second image of the spectrum of a sampled signal, which is derived from a model involving Dirac deltas.

An alternative view on this problem is presented in (Borys A. 2020a). It is based on consideration of the first possible representation of the sampled signal spectrum (i.e. without involvement of Dirac deltas) and leads to quite different conclusions.

The objective of this paper is to show, from another perspective, that really the spectrum of a sampled signal (sampled in an ideal way) cannot be uniquely defined. And, as it does not exist (Borys A. 2020a), (Borys A. 2020b) as a Fourier transform of a true sampled signal, an extension of its definition is needed, what can be done in a variety of ways – as proposed, for example, in (Borys A. 2020b). So, this takes place in an arbitrary way.

To strengthen this point of view, that is arbitrariness of the choice mentioned above, we consider here an example of solving a problem, in which two definitions of the spectrum of a sampled (ideally) signal are tacitly used, but not named explicitly at all. (Note that if these different definitions of the same object were used this would be viewed as a mistake.)

Our example is the Shannon's proof of the reconstruction formula (Shannon C. E. 1949) applied, here, for obtaining a description of the signal sampling and sampled signal spectrum. It is presented and analyzed in the next section. The paper ends with a concluding remark.

2 SHANNON'S PROOF OF RECONSTRUCTION FORMULA APPLIED TO DESCRIPTION OF SIGNAL SAMPLING AND SAMPLED SIGNAL SPECTRUM

In this section, we consider a case when a signal $y(t)$ of a continuous time t is a bandlimited one. And, we denote the maximal frequency present in its spectrum by f_m. So, this signal can be sampled and reconstructed perfectly if the sampling period T fulfils the following Nyquist-Shannon condition:

$$1/T = f_s \geq 2 f_m, \tag{2}$$

where f_s means the corresponding sampling frequency. (Note that from now the symbol T will play here two roles: of a signal repetition time and of a signal sampling period – the one applicable at the moment will follow from the context; moreover, the sampling frequency f_s corresponds with the frequency f_0 that was defined and used in the previous section.)

It follows from the above that the Fourier transform $Y(f)$ of the signal $y(t)$ has nonzero values only on the segment $<-f_m, f_m>$ of the frequency axis (that is supported only on this segment). This property allows to expand it on the whole frequency axis – in form of a Fourier series.

Obviously, one can take into account a wider range of frequencies around $Y(f)$ than $<-f_m, f_m>$, and treat it as "an extended support" of $Y(f)$. And, just this is done in what follows – it makes a slight modification of the Shannon's scheme in (Shannon C. E. 1949). We build up a periodic function $Y_p(f)$ from $Y(f)$ on the following "extended supporting interval": $<-f_s/2, f_s/2>$ with f_s given by (2). In other words, we perform here a periodization of $Y(f)$ on its "extended support" to get a periodic function on the whole frequency axis.

Assume now that the function $Y_p(f)$ so obtained fulfills the conditions (Bracewell R. N. 2000), (Brigola R. 2013) allowing its expansion in a Fourier series. That is we get then

$$\begin{aligned} Y_p(f) &= \sum_{k=-\infty}^{\infty} a_k \exp(j2\pi kTf) = \\ &= \sum_{k=-\infty}^{\infty} a_k \exp(j2\pi k\, f/f_s) \end{aligned}, \tag{3}$$

where the coefficients $a_k = a(k)$ are given by

$$a_k = \frac{1}{f_s} \int_{-f_s/2}^{f_s/2} Y_p(f) \exp(-j2\pi k\, f/f_s)\, df \,. \tag{4}$$

In the next step, observe that (4) can be re-written as

$$\begin{aligned} a_k &= \frac{1}{f_s}\Bigg[\int_{-f_s/2}^{-f_m} Y_p(f)\exp(-j2\pi k\, f/f_s)\,df + \\ &+ \int_{-f_m}^{f_m} Y_p(f)\exp(-j2\pi k\, f/f_s)\,df + \\ &+ \int_{f_m}^{f_s/2} Y_p(f)\exp(-j2\pi k\, f/f_s)\,df \Bigg] = \\ &= \frac{1}{f_s}\int_{-\infty}^{\infty} Y(f)\exp(-j2\pi k\, f/f_s)\,df \,. \end{aligned} \tag{5}$$

The final result in (5) follows from the fact that the first and third integrals there are equal to zero, $Y_p(f) \equiv Y(f)$ in the interval $<-f_m, f_m>$, and $Y(f)$ is identically zero outside the latter frequency range.

Further, note that the result achieved in (5) can be also expressed in the following way:

$$\begin{aligned} a_k &= \frac{1}{f_s}\int_{-\infty}^{\infty} Y(f)\exp\big(j2\pi(-k/f_s)\, f\big)\,df = \\ &= T\int_{-\infty}^{\infty} Y(f)\exp\big(j2\pi(-kT)\, f\big)\,df \,, \end{aligned} \tag{6}$$

and that this is a form of the inverse Fourier transform of $Y(f)$ calculated at the time point $-kT$. So, it allows us to write

$$a_k = T \cdot y(-kT). \tag{7}$$

And substituting (7) into (3) gives

$$Y_p(f) = T\sum_{k=-\infty}^{\infty} y(-kT)\exp(j2\pi kTf) \cdot \tag{8}$$

Note now that (8) can be rewritten in form of the so-called discrete time Fourier transform (DTFT) (McClellan J. H., Schafer R., Yoder M. 2015), (Vetterli M., Kovacevic J., Goyal V. K. 2014), (Wang R. 2010), (Ingle V. K., Proakis J. G. 2012), (Oppenheim A. V., Willsky S., Nawab S. H. 1996) of the discrete signal $y(kT)$ – which is obtained by sampling the signal $y(t)$ with the rate $f_s = 1/T$ and which shows only the samples of $y(t)$ (that is without "any interest in what happens in the intervals between the successive moments of sampling"). To see this, let us introduce an auxiliary index $k' = -k$ in (8). This results in

$$\begin{aligned} Y_p(f) &= T\sum_{k'=\infty}^{-\infty} y(k'T)\exp(-j2\pi k'Tf) = \\ &= T\sum_{k'=-\infty}^{\infty} y(k'T)\exp(-j2\pi k'Tf)\,. \end{aligned} \tag{9}$$

Omitting afterwards the prime symbol at k' in (9), we obtain

$$Y_p(f) = \sum_{k=-\infty}^{\infty} \bar{y}(kT)\exp(-j2\pi kTf) \tag{10}$$

with $\bar{y}(kT) = y(kT)\cdot T$. So, finally, we see that the right-hand side of (10) constitutes really a definition of the DTFT of the signal $y(kT)$, see (McClellan J. H., Schafer R., Yoder M. 2015), (Vetterli M., Kovacevic J., Goyal V. K. 2014), (Wang R. 2010), (Ingle V. K., Proakis J. G. 2012), (Oppenheim A. V., Willsky S., Nawab S. H. 1996).

On the other hand, we know from the literature (Marks II R. J. 1991), (Vetterli M., Kovacevic J., Goyal V. K. 2014), (Oppenheim A. V., Schafer R. W., Buck J. R. 1998), (Bracewell R. N. 2000), (McClellan J. H., Schafer R., Yoder M. 2015), (Brigola R. 2013), (So H. C. 2019), (Wang R. 2010), (Ingle V. K., Proakis J. G. 2012), (Jenkins W. K. 2009) that $Y_p(f)$ is also identified, at the same time, with the spectrum (i.e. called the spectrum) of the sampled signal, say $y_{D,T}(t)$, modelled as a generalized function of a continuous time *t* and consisting of the discrete signal $y(kT)$ mentioned just before with zeros filling the intervals between the successive points of sampling. So, let us express this fact as

$$\mathrm{SPECT1}(y_{D,T}(t)) = Y_p(f) = \mathrm{DTFT}(y(kT)) \;, \tag{11}$$

where $\mathrm{SPECT1}(y_{D,T}(t))$ denotes one of the possible definitions of the spectrum of the sampled signal $y_{D,T}(t)$ that uses the notion of DTFT in the sense as explained above.

As well known, the identity between $\mathrm{SPECT1}(y_{D,T}(t))$ and $Y_p(f)$ is also manifested in the literature (Marks II R. J. 1991), (Vetterli M., Kovacevic J., Goyal V. K. 2014), (Oppenheim A. V., Schafer R. W., Buck J. R. 1998), (Bracewell R. N. 2000), (McClellan J. H., Schafer R., Yoder M. 2015), (Brigola R. 2013), (So H. C. 2019), (Wang R. 2010), (Ingle V. K., Proakis J. G. 2012), (Jenkins W. K. 2009) in another way, namely by writing the following:

$$\begin{aligned}\mathrm{SPECT1}(y_{D,T}(t)) &= \frac{1}{T}\sum_{k=-\infty}^{\infty} Y(f - k/T) = \\ &= Y_p(f) \;.\end{aligned} \tag{12}$$

Observe now that the derivations and relationships presented above suggest that, probably, the expression on the right-hand side of (10) has been named the DTFT because it incorporates samples of an analog signal, a sum replacing an integral, and exponential functions of the type: $\exp(-j2\pi kTf)$ – all of them connected with each other into a whole resembling an usual Fourier transform. And, as shown in (10), when we go from the right to the left there, this DTFT equals the auxiliary periodic function $Y_p(f)$ calculated in the Shannon's proof. But, the Shannon's proof does not need to define the spectrum of the sampled signal $y_{D,T}(t)$.

Let us examine however correctness of the definition of the spectrum $\mathrm{SPECT1}(y_{D,T}(t))$ assumed in (11). To this end, assume for a moment that there exists an inverse operator, say $\mathrm{SPECT1}^{-1}$, which enables to obtain the sampled signal $y_{D,T}(t)$ from its spectrum $\mathrm{SPECT1}(y_{D,T}(t))$. Formula (12) tells us how it could look like, namely as

$$\begin{aligned}y_{D,T}(t) &= \frac{1}{T}\mathcal{F}^{-1}\left(\sum_{k=-\infty}^{\infty} Y(f - k/T)\right) = \\ &= \frac{1}{T}\sum_{k=-\infty}^{\infty} \mathcal{F}^{-1}\left(Y(f - k/T)\right) \;,\end{aligned} \tag{13}$$

where $\mathcal{F}^{-1}(\cdot)$ stands for the usual inverse Fourier transform. Further, it is easy to obtain, from (13), the following:

$$y_{D,T}(t) = y(t)\frac{1}{T}\sum_{k=-\infty}^{\infty}\exp\left(j2\pi(k/T)t\right) \tag{14}$$

However, note now that it has been shown in (Borys A. 2020c) that the operations performed in (13) cannot be considered as being fully correct within the classic mathematics (i.e. the one which does not include such objects like distributions, in particular Dirac distributions (Dirac P. A. M. 1947)). The reason of this, detailed explanations, and a remedy to circumvent the problem have been provided in (Borys A. 2020c). This material will not be, however, repeated here because of a lack of space as well as to avoid accusation of auto-plagiarism. Moreover, the reference (Borys A. 2020c) is well available.

In what follows below, we use the main result from (Borys A. 2020c); it says that the definition of the DTFT occurring on the right-hand side of (10) must be modified to

$$\begin{aligned}&\mathrm{DTFTm}(y(kT)) = \sum_{k=-\infty}^{\infty} \bar{y}(kT)\exp(-j2\pi kTf) \\ &\quad \text{for} \quad |Tf = f/f_s| \le 1/2 \quad \text{and} \\ &\mathrm{DTFTm}(y(kT)) \equiv 0 \quad \text{for} \quad |Tf = f/f_s| > 1/2 \;,\end{aligned} \tag{15}$$

where $\mathrm{DTFTm}(\cdot)$ stands for the modified DTFT after the theory presented in (Borys A. 2020c). Therefore, the middle expression in (12), expressing the DTFT in an equivalent way, must be modified, too. Then, it has the following form:

$$\mathrm{DTFTm}(y(kT)) = Y(f) \;. \tag{16}$$

For details of derivation of (16), see (Borys A. 2020c).

Along the same lines as before, let us now identify it with the spectrum of a sampled signal. That is, let us write an equivalent of (11) for this case. We get then

$$\mathrm{SPECT2}(y_{E,T}(t)) = Y(f) = \mathrm{DTFTm}(y(kT)) \;, \tag{17}$$

where $\mathrm{SPECT2}(y_{K,T}(t))$ stands for another possible definition of the spectrum (which exploits the notion of the DTFTm), and the sampled signal is denoted now by $y_{E,T}(t)$. Further, application of the inverse Fourier transform in (17) gives

$$y_{E,T}(t) = y(t) = \mathcal{F}^{-1}\left(Y(f)\right). \tag{18}$$

Now, let us "demonstrate the occurrence of samples " in these two signals $y_{D,T}(t)$ and $y_{E,T}(t)$ that model the sampled signal in the continuous time domain. To this end, see that the sum of exponentials in (14) can be expressed as the so-called Dirac comb multiplied by T (Bracewell R. N. 2000), (Osgood B. 2014). So, this gives

$$y_{D,T}(t) = y(t)\cdot\frac{T}{T}\mathrm{comb}_T(t) = \\ = \sum_{k=-\infty}^{\infty} y(kT)\delta(t-kT)\ , \tag{19}$$

where the symbol $\mathrm{comb}_T(t)$ stands for the Dirac comb (Bracewell R. N. 2000), (Osgood B. 2014). Further, for $y_{E,T}(t)$, we need to use the reconstruction formula (Marks II R. J. 1991), (Vetterli M., Kovacevic J., Goyal V. K. 2014), (Oppenheim A. V., Schafer R. W., Buck J. R. 1998), (Bracewell R. N. 2000), (McClellan J. H., Schafer R., Yoder M. 2015), (Brigola R. 2013), (So H. C. 2019), (Wang R. 2010), (Ingle V. K., Proakis J. G. 2012), (Jenkins W. K. 2009). Applying it in (18), we arrive at

$$y_{E,T}(t) = \sum_{k=-\infty}^{\infty} y(kT)\,\mathrm{sinc}(t/T-k)\ , \tag{20}$$

with the function $\mathrm{sinc}(x)$ defined as $\mathrm{sinc}(x) = \sin(\pi x)/\pi x$ for $x \neq 0$ and 1 for $x = 0$.

Observe now that both the signals $y_{D,T}(t)$ and $y_{E,T}(t)$ differ from a true image of the sampled signal considered in the continuous time domain (Borys A. 2020b). (This image is called in (Borys A. 2020b) a reference representation of the sampled signal and is denoted by $x_{R,T}(t)$ there.) So, for getting $x_{R,T}(t)$ from $y_{D,T}(t)$ or $y_{E,T}(t)$, an additional operation is needed (see for more details (Borys A. 2020b)). In other words, our conclusion at this point is that the true sampled signal $x_{R,T}(t)$ cannot be obtained neither by an inverse operation of the spectrum $\mathrm{SPECT1}(\cdot)$ nor by an inverse operation of the spectrum $\mathrm{SPECT2}(\cdot)$.

Let us now come back to the Shannon's proof (Shannon C. E. 1949), to its second part (which can be loosely understood as obtaining the continuous time domain version of a signal from its discrete (digital) form). In short, it can be expressed in the following way:

$$y(t) = \int_{-\infty}^{\infty} Y(f)\exp(j2\pi ft)df = \\ \int_{-f_s/2}^{f_s/2} Y_p(f)\exp(j2\pi ft)df = \\ = \int_{-f_s/2}^{f_s/2} \mathrm{DTFT}\left(y(kT)\right)\exp(j2\pi ft)df = \\ \int_{-\infty}^{\infty} \mathrm{DTFTm}\left(y(kT)\right)\exp(j2\pi ft)df = \\ = \int_{-f_s/2}^{f_s/2}\left(\sum_{k=-\infty}^{\infty}\bar{y}(kT)\exp(-j2\pi kTf)\right)\exp(j2\pi ft)\cdot \\ \cdot df = \sum_{k=-\infty}^{\infty}\bar{y}(kT)\int_{-f_s/2}^{f_s/2}\exp\left(-j2\pi f(kT-t)\right)df = \\ = \sum_{k=-\infty}^{\infty} y(kT)\,\mathrm{sinc}(t/T-k)\ . \tag{21}$$

3 CONCLUDING REMARK

As a concluding remark, let us observe in (21) that both the spectra of the two possible definitions of the sampled signal occur implicitly in the Shannon's proof of the reconstruction formula, but they are absolutely superfluous there. So, this underlines their arbitrariness, artificialness, and rather a limited usefulness – as we tried to show in this paper.

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