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Behavior of Formula for Spectrum of Sampled Signal when Sampling Period Tends to Zero

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ABSTRACT: This paper is devoted to consideration of behaviour of two formulas used in the literature to describe the spectra of sampled signals in the case of value of the sampling period going to zero. It is shown here that they do not lead to the same result, as we would expect. Only one of them gives an outcome compatible with reality.

1 INTRODUCTION

Analog signal sampling is an operation performed on a signal (function) of a continuous time *t*. Its purpose is to acquire a sequence of samples of this signal for performing, for instance, a digital filtering with the use of a signal processor. The operation of sampling is performed by A/D (analog-to-digital) signal converters. But the signal at the output of these electronic devices can be viewed in two ways. First, as a discrete function with the values of its dependent variable being the sample values taken at signal sampling instants, and its independent variable running through the indices (from a set of integers) of the time instants mentioned. That is it can be perceived as a signal in which the physical time *t* is not important; here, only the information about the order in which the signal samples occur is needed. And this form of the signal is used in digital signal processing. But, secondly, the signal at the output of an A/D converter can be also viewed as a sampled signal that is a function (or object) that depends upon the physical time *t*. Then it represents a function (object) with an independent variable *t*, which means a continuous time (whose values belong to the set of real numbers).

Note now that the Fourier transform (spectrum) of a sampled signal, which we have at the output of an A/D converter (and which we view as a function of a continuous time t), can be calculated according to the classical definition, i.e. as a Fourier integral. Or if we perceive a sampled signal as an object called the weighted Dirac comb (with an independent variable t), its Fourier transform can be obtained via application of the corresponding rules that are used in the case of Dirac distributions.

In the first case, in which one decides to describe the waveform of a continuous time at the output of an A/D converter as a function of this time, the best description of it is via a step function [1], [2]. This is the best way when we model the sampling operation as a one that is performed perfectly (ideally). Then the waveform mentioned above and denoted below by $x_{STEP}(t)$ is expressed as

$$xSTEP(t) = \sum_{k=-\infty}^{\infty} x(kT)\, rect(t-kT) \qquad (1)$$

where $x(t)$ is an analog signal applied at the input of an A/D converter. Further, $x(kT)$'s in (1) with k's belonging to the set $\mathbb{Z}$ of integers stand for its samples. And T means the period of sampling. Finally, $rect(\cdot)$ in (1) is a standard rectangular function that is given by the following formula:

$$rect(t) = 1 \quad for \quad t \in \langle 0, T) \quad and\ 0\ otherwise \tag{2}$$

It has been shown [2] that the Fourier transform, i.e. the spectrum of the function (1), is given by

$$X_{STEP}(f) = \left(\sum_{k=-\infty}^{\infty} X(f - kf_s) \right) \cdot \operatorname{sinc}(\pi fT) \exp(-j\pi fT) \tag{3}$$

where $X_{STEP}(f)$ stands for the Fourier transform of $x_{STEP}(t)$ and $X(f\text{-}kf_s)$'s are the frequency-shifted Fourier transforms of $x(t)$. Furthermore, f_s=1/T is the sampling frequency, $j = \sqrt{-1}$, and sinc(·) means the standard function defined as

$$\operatorname{sinc}(x) = \begin{cases} \dfrac{\sin(x)}{x} & for \quad |x| \neq 0 \\ 1 & for \quad x = 0 \end{cases} \tag{4}$$

Consider now the second approach [3]–[15] to modelling the waveform of a continuous time at the A/D converter output with the use of the weighted Dirac comb. Then this waveform, denoted here by x_D(t), is expressed as

$$x_D(t) = \delta_T(t) \cdot x(t) = \sum_{k=-\infty}^{\infty} x(kT)\delta(t - kT) \tag{5}$$

where the object $\delta_T(t)$, which is dependent upon time t, is called the Dirac comb and defined by

$$\delta_T = \sum_{k=-\infty}^{\infty} \delta(t - kT) \tag{6}$$

In (6), $\delta(t\text{-}kT)$'s, $k \in \mathbb{Z}$, mean the time-shifted Dirac deltas.

It has been shown in the literature [3]–[15] that the Fourier transform of (5), i.e. its spectrum, is given by

$$X_D(f) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(f - kf_s) \tag{7}$$

where $X_D(f)$ stands for the Fourier transform of $x_D(t)$.

Formulas (3) and (7) for the spectrum of a sampled signal, i.e. of a waveform at the output of a sampling device differ obviously from each other because they are obtained with the use of different models (descriptions) of the sampling (ideal) process. To determine which one better describes reality, it seems worth checking how they behave in a limiting case, for example, when the value of the sampling period T goes to zero. Do they converge then to the same form? It seems worth doing such a 'test' because it is justified and for a very simple reason. When the sampling period goes to zero, the time waveform observed at the output of the sampling device becomes more and more similar to that applied to its input. Hence, the spectrum of the former signal should become closer and closer to the spectrum of the latter one, too. Does it really happen? This question is answered in the next section. The paper ends with a conclusion.

2 COMPARISON OF SPECTRA OF SAMPLED SIGNAL VIA (3) AND (7) WHEN SAMPLING PERIOD TENDS TO ZERO

Let us perform here the test mentioned in the previous section. To perform it, assume that we are interested only in the frequency range which we are able to observe physically (that is, for example, carry out in it a spectral analysis of the sampled signal). Moreover, we restrict ourselves here to considering sampling of only band-limited signals. Then, as well known, if the Nyquist condition of sampling [3]–[15] is satisfied, aliasing does not occur. As a consequence of this, if the sampling frequency f_s is high enough, that is the sampling period T is small enough, then only one 'nonzero pattern' of the spectrum of the signal $x(t)$ occurs in the range of frequencies mentioned above. In other words, we can express this fact as follows: then only one 'nonzero pattern' of the spectrum of the signal $x(t)$ is 'visible' to us (for the range of frequencies mentioned); all the other 'nonzero patterns' are outside this range. And, here, we denote it by f_{ob}.

It is best to illustrate the above with a suitable example. The author of this paper has already come up with such an example and discussed it in one of his previous publications [16]. So there is nothing left but to use it here as well; we exploit it in what follows. In this example, we refer to the radio frequency (RF) range, i.e. the frequencies of electromagnetic waves that are used in radio communication. It is assumed that these are the frequencies whose scope extends from about 3 kHz to about 3 THz. Therefore, the maximum range of frequencies covered by a "RF spectrum nonzero pattern" on the frequency axis will be equal to approximately $2f_{ob}$ = 6 THz (from $-f_{ob}$ to $+f_{ob}$). Further, by identifying f_{ob} with f_m, i.e. the maximal frequency present in the spectrum of the signal $x(t)$, occurring in the Nyquist condition

$$1/T = f_s \geq 2 f_m \tag{8}$$

for the absence of aliasing effects in the sampled signal spectrum [3]–[15], we get $T \leq 1/(2f_{ob})$ from the above inequality. And finally, substituting the value of $2f_{ob}$ = 6 THz mentioned before into the latter inequality, we obtain T ≤ 0,16 ps = 160 fs.

So if we sample any RF signal with the sampling periods smaller or equal to 160 fs, we will not experience any periodicity of the sampled signal spectrum. Because, then, this periodicity will be not 'visible' in the observed range of frequencies extending from 0 Hz to 3 THz.

Thus, based on this example, the formulas (3) and (7) can be rewritten to the following forms:

$$X_{STEPob}(f) = X(f) \operatorname{sinc}(\pi fT) \exp(-j\pi fT), \tag{9}$$

and

$$X_{Dob}(f) = \frac{X(f)}{T} \tag{10}$$

for RF signals when they are sampled with sampling periods smaller or equal to 160 fs. In (9) and (10), $X_{STEPob}(f)$ and $X_{Dob}(f)$ mean, respectively, $X_{STEP}(f)$ and $X_D(f)$ valid for just this range of sampling periods.

In the next step, note that the following relations:

$$\lim_{T \to 0} \left(X_{STEPob}\left(f \right) \right) = X\left(f \right) \tag{11}$$

and

$$\lim_{T \to 0} \left(X_{Dob}\left(f \right) \right) = \infty \quad \textit{for all } X\left(f \right) \neq 0 \tag{12}$$

hold.

Comparison of (11) with (12) shows that the limits indicated there do not converge to the same result. Hence, it should be concluded that the modelling of the signal sampled via a sequence of weighted Dirac deltas differs (we can say significantly) from the modelling of this signal by means of a step waveform. However, we note that the limit in (11) is consistent with our expectation because in the case of decreasing the value of the sampling period (T tending to zero), the sampled signal coincides more and more with the signal $x(t)$, whose spectrum is just $X(f)$. So we must therefore conclude that modelling the sampled signal with a step waveform is closer to reality.

3 CONCLUSION

This paper presents another (new) result demonstrating the weakness of the approach, in which the sampled signals are described via sequences of the weighted Dirac deltas.

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