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# Approximation Models of Orthodromic Navigation

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ABSTRACT: The paper deals with two different approaches to orthodromic navigation approximation, the secant method and the tangent method. Two ways of determination of orthodromic interposition coordinates will be presented with the secant method. In the second, tangent method unit change of orthodromic course (ΔK) will be used.

## 1 INTRODUCTION

Navigation on the surface of the Earth is possible in two ways: by orthodrome and loxodrome. Orthodrome is a minor arc of the great circle bounded by two positions, and corresponds to their distance on a surface of the Earth, representing also the shortest distance between these positions on the Earth as a sphere. The ship, travelling in orthodromic oceanic navigation, has her bow directed towards the port of arrival all the time. The orthodorme is the curve of a variable course – it intersects meridians at different angles. When navigating by the orthodrome, course should be constantly changed, which is unacceptable from the navigational point of view. On the other hand, loxodrome (rhumb line) intersects all meridians at the same angle, and it is more suitable in maintaining the course. However, loxodromic path is longer that the orthodromic one. Sailing by loxodrome, the bow of the ship will be directed toward the final destination just before arrival. Due to the mentioned facts, it is necessary to use the advantages of both curves – the shorter path of the orthodrome and the rhumb line conformity.

Orthodrome navigation is, as mentioned, inconvenient. Therefore, only approximation of orthodrome navigation can be taken into account, reducing the number of course changes to an acceptable number – always bearing in mind that if the number of course alteration is greater, the navigation is closer to the great circle. After defining elements for course and distance determination on an orthodrome curve, navigation between the derived points is carried out in loxodromic courses.

Applying spherical trigonometry, the proposed paper elaborates models of approximation for the orthodrome navigation with the secant method and the tangent method. The secant method provides two models of navigation. In the first model, the orthodrome is divided into desired waypoints – interpositions between which the ship sails in loxodromic courses. The second model of the method implies the path between two positions divided into specific intervals of unit distances, which then define other elements of navigation (interposition coordinates and loxodromic courses). In these two models, navigation has been approximated with the secants of the orthodrome curve on which the vessel sails. The tangent method gives an approximation model by determining the unit changes of orthodromic courses, and defining the tangent on a curve, after which other navigational elements needed for navigation are performed.

## 2 IMPORTANT RELATIONS BETWEEN ORTHODROMIC AND LOXODROMIC DISTANCES FOR THE EARTH AS A SPHERE

As described above, the rhumb line, i.e. loxodrome, represents a constant course, spiral-shaped curve, asymptotically approaching the Pole. The orthodrome represents a variable course curve, the minor arc of the great circle between two positions. For the Earth as a sphere, between positions $P_1$ and $P_2$, these two distances are equal in two situations only (Figure 1):

1 if the positions are placed on the same meridian, then $\Delta\lambda=0$, $\Delta\varphi\neq0$,

2 if the positions are placed on the Terrestrial Equator, then $\Delta\varphi=0$, $\Delta\lambda\neq0$

In all other situations, the orthodrome distance is always smaller, or $D_O \neq D_L$.

Maximum difference between distances $D_O$ and $D_L$ occurs when $\varphi_1 = \varphi_2 \neq 0$, $\Delta\varphi = 0$, $\Delta\lambda = 180^\circ$ between $P_1$ and $P_2$ is applied. In this case, the function extremum should be determined [Wippern, 1992]:

$$f_{(\varphi)} = \Delta\lambda \cos\varphi - 2\left(90^\circ - \varphi\right)$$

$$f_{(\varphi)} = \Delta\lambda \cos\varphi - \pi + 2\varphi$$

$$f'_{(\varphi)} = -\Delta\lambda \sin\varphi + 2$$

$$f''_{(\varphi)} = -\Delta\lambda \cos\varphi < 0$$

For φ from 0º to ± 90º the cosφ function is positive, so the second derivation $f'_{(\varphi)}<0$. Therefore, the function has an extremum, maximum:

$$f'_{(\varphi)} = 0 - \Delta\lambda \sin\varphi + 2 = 0$$

$$\varphi = \sin^{-1}\left(\frac{2}{\Delta\lambda}\right)$$

$$f_{(\varphi)}'' = -\Delta\lambda \cos\varphi = -\Delta\lambda\sqrt{1-\sin^2\varphi}$$

$$f'_{(\varphi)} = -\Delta\lambda\sqrt{1-\frac{4}{\Delta\lambda^2}} > 0$$

If $\Delta\lambda = 180^\circ = \pi$, then $\varphi = 39^\circ\,32'\,24{,}8''$

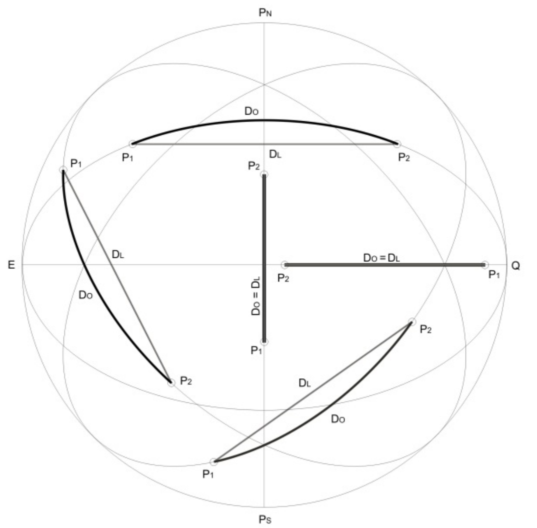


Figure 1: Orthodrome and Loxodrome relations on Earth as a sphere
Source: Made by authors

From the completely theoretical point of view, it follows that the strongest difference between orthodromic and loxodromic distance appears if positions $P_1$ and $P_2$ are placed on a geographic parallel of $\varphi = 39^\circ\,32'\,24{,}8''$, and on anti-meridians, i.e. where $\Delta\lambda = 180^\circ$. Then, the function value $f_{(\varphi)}$ amounts 2273, 5475...M [Benković et al]; this value represents the difference between orthodromic and loxodromic path. Maximum numerical saving of 2273,5 M in the distance, expressed in percentage counts 37,5%[1]. In most cases, the distance saving in percentages in navigational practice reachs up to 10%.

In Equator/Meridian sailing, as well as heading close to the corresponding courses, the distance saving is minimal, given that the curves are more and more closer. In other cases, that are headings in the 090º/270º sector, particularly when sailing on the same parallel (with appropriate distance between positions), approximating the navigation could save up to one day of navigation, which nowadays represents an important element of the navigation venture.

## 3 APPROXIMATION OF ORTHODROMIC NAVIGATION BY SECANT METHODS

### 3.1 *The first secant method – Orthodrome interposition division*

In the first secant method the problem is approached in a way that the orthodrome is divided into interpositions, between which the vessel sails in loxodromic courses.

Interpositions differ in their longitude every 5º or 10º (mostly), while the division begins from the Vertex of the orthodrome, under the condition that this point is placed between the departure and arrival position. If Vertex is situated outside of the specific positions, interpositions can be defined from the point of departure, $P_1$. In the following text, Vertex interposition division is explained. In Figure 2. the required relations between the elements are shown.

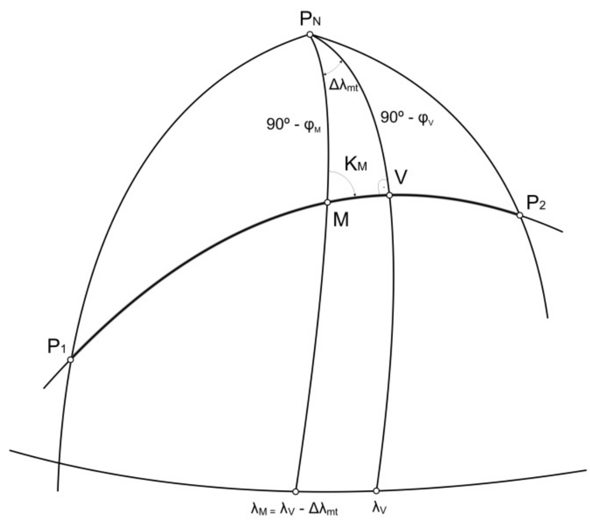


Figure 2: Determining the coordinates of orthodrome interpositions
Source: Made by authors

[1] Theoretically speaking, expressed saving can reach up to 57%, but it has no practical importance for navigation, because these are very short paths between positions on the parallel at the near Pole (e.g. φ=88º).

V – Vertex of the orthodrome, defined by the coordinates $\varphi_V$ i $\lambda_V$

$\Delta\lambda_{mt}$– the selected difference of longitude for which interpositions are required

M– orthodrome interposition, defined by the coordinates $\varphi_M$ i $\lambda_M$

The navigator selects the interposition longitude:

$$\lambda_M = \lambda_V - \lambda_{mt}$$

The latitude is obtained by applying spherical trigonometry for the right-angled triangle $\Delta P_N MV$ [Kos et al, 2010]:

$$\cos\Delta\lambda_{mt} = \mathrm{ctg}\,[90° - (90° - \varphi_V)]\,\mathrm{ctg}\,(90° - \varphi_M)$$
$$\cos\Delta\lambda_{mt} = \mathrm{ctg}\varphi_V\ \mathrm{tg}\varphi_M$$
$$\mathrm{tg}\varphi_M = \cos\Delta\lambda_{mt}\ \mathrm{tg}\varphi_V$$
$$\varphi_M = \mathrm{arc\,tg}\,(\cos\Delta\lambda_{mt}\ \mathrm{tg}\varphi_V)$$

In case that the Vertex lies outside positions $P_1$ and $P_2$, the division begins from the point $P_1$. Here, the inclination of the orthodrome (*i*) should be determined first. The inclination of orthodrome represents the angle at which orthodrome intersects the Equator of the Earth, resulting in a right triangle of the point of departure, $P_1$.

$$\cos i = \sin\,(90° - \varphi_1)\sin\alpha$$
$$\cos i = \cos\varphi_1\ \sin\alpha$$
$$i = \arcsin\,(\cos\varphi_1\ \sin\alpha)$$

The longitude of the intersection, $\lambda_S$, is defined as follows:

$$\lambda_S = \lambda_1 + \Delta\lambda_S \quad \text{where}$$
$$\mathrm{tg}\Delta\lambda_S = -\sin\varphi_1\ \mathrm{tg}\alpha$$
$$\Delta\lambda_S = \mathrm{arc\,tg}\,(-\sin\varphi_1\ \mathrm{tg}\alpha)$$ [2]

In a right-angled triangle $\Delta SAM$, equatorial leg ($\Delta\lambda_S + \Delta\lambda_{mt}$) and the angle of inclination *i* are known. The following relation are a result of this triangle (Figure 3) [Kos et al, 2010]:

$$\cos\,[90° - (\Delta\lambda_S + \Delta\lambda_{mt})] = \mathrm{ctg}\,i\ \mathrm{ctg}\,(90° - \varphi_M)$$
$$\sin\,(\Delta\lambda_S + \Delta\lambda_{mt}) = \mathrm{ctg}\,i\ \mathrm{tg}\varphi_M$$
$$\mathrm{tg}\varphi_M = \sin\,(\Delta\lambda_S + \Delta\lambda_{mt})\ \mathrm{tg}\,i$$
$$\varphi_M = \mathrm{arc\,tg}\,[\sin\,(\Delta\lambda_S + \Delta\lambda_{mt})\ \mathrm{tg}\,i]$$

[2] Some of mentioned elements perhaps require additional explanation, mathematical derivation respectively. Bearing in mind the length limitation of the paper, as well as the extensive nature of the matter, the reader is referred to the additional literature [Kos et al, 2010].

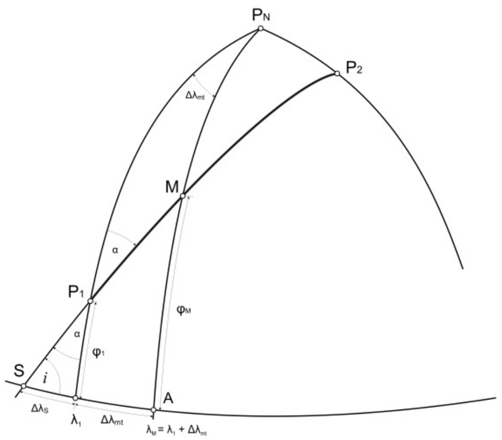


Figure 3: The inclination of the orthodrome
Source: Made by authors

#### 3.1.1 *Loxodromic intercourse and distances determination*

The loxodromic courses between the positions are calculated from the loxodromic triangle [Benković et al, 1986]:

$$\mathrm{tgK} = \frac{\Delta\lambda}{\Delta\varphi_M}$$
$$\mathrm{K} = \mathrm{arc\,tg}\left(\frac{\Delta\lambda}{\Delta\varphi_M}\right)$$

The first course ($K_1$), by which the orthodrome navigation begins (in position $P_1$), is calculated on the basis of $\Delta\lambda$, that is, the longitude difference between $P_1$ and the first interposition, $M_1$, and the Mercator latitudes difference between the same points. The second course ($K_2$) in $M_2$ is calculated on the basis of analogic $\Delta\lambda$ and $\Delta\varphi_M$ points $M_1$ and $M_2$, etc. The Figure 4. shows graphic determination of loxodromic courses and distances.

Besides loxodromic courses between two interpositions, to determine the distances, one needs to know the latitude difference between the positions, beginning at $P_1$ and $M_1$, then $M_1$ and $M_2$ and so on to the point of arrival $P_2$;

$$D_L = \frac{\Delta\varphi}{\cos K}$$

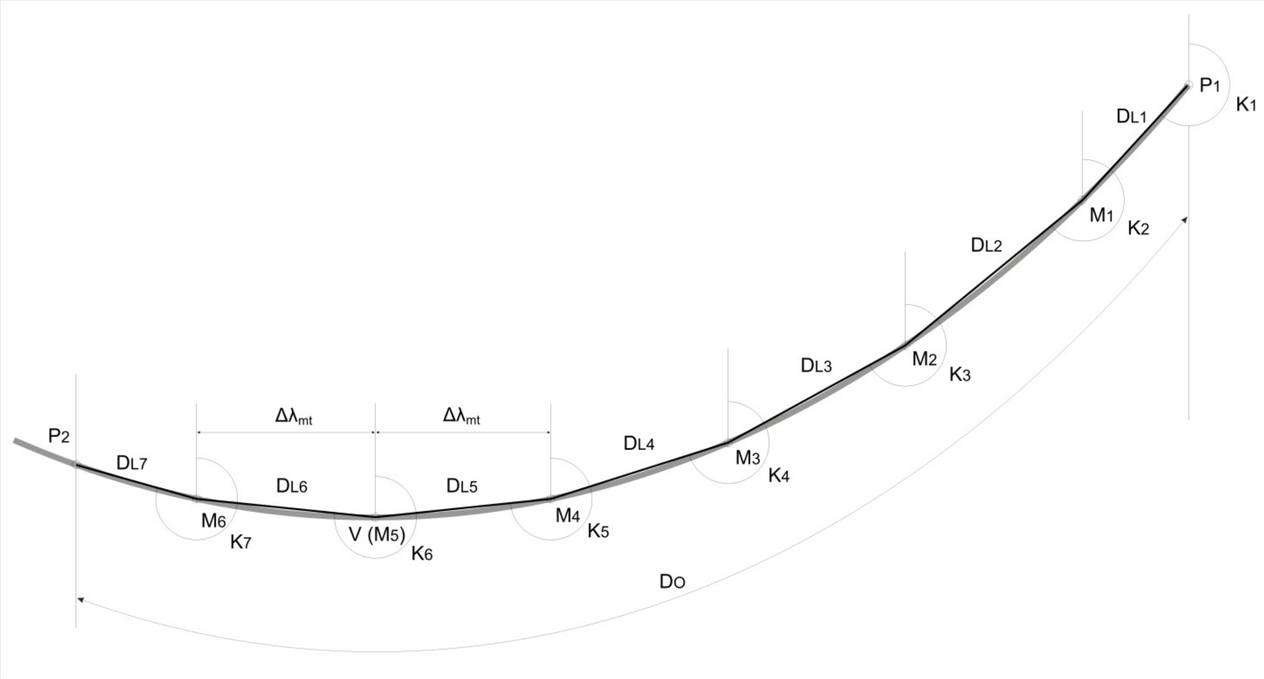


Figure 4: Orthodromic navigation approximation by the interposition division – The first secant method

Source: Made by authors

### 3.2 *The second secant method – Division of the orthodrome in unit distance intervals*

This method is based on the theoretical assumption that the orthodrome, which passes through two positions on the surface of the Earth as a sphere, is composed of an infinite number of infinitively small loxodromes [Kos, 1996], i.e.

$$D_O = \int_O^L \Delta d_L$$

It follows that the final greatness of the orthodrome passing through two positions that are sufficiently distant from each other, can be replaced with the infinitivelly small number of loxodromes, i.e. $dD_L = dD_O$, respectively:

$$D_O = \int_O^{D_O} dD_O$$

Given that the greatness of infinitively small loxodrome cannot be dimensionally defined, the loxodrome could be defined by the approximation of the greatness of orthodromic unit distance intervals ($dD_O$), which is then approximately equal with the loxodromic distance ($dD_L$). In this way, the inconvenient orthodrome navigation is replaced with the loxodrome sailing. The intention is that the course alternations are reduced to a navigationally acceptable amount. The smaller the greatness of orthodromic unit distance, the minor the error of orthodromic approximation. However, it requires more frequent course alternation, which is in contradiction with practical navigation. Therefore, it is proposed as follows:

- if two positions on Earth (approximated by the shape of the sphere) are distant one from another ≤ 30' = 30 M, the following approximation can be introduced:

$DL \cong DO = 30'$

Based on the above mentioned, the concept of unit distance interval is introduced, and it is 30', i.e. 30 M.

The process of orthodromic navigation performing is as follows:

$P_1$ ($\varphi_1$, $\lambda_1$) – departure position coordinates

$P_2$ ($\varphi_2$, $\lambda_2$) – arrival position coordinates

Orthodromic distance between $P_1$ and $P_2$ is calculated, using the equation which is derived from the nautical – positioning spherical triangle:

$$D_O = \cos^{-1}(\sin\varphi_1 \sin\varphi_2 + \cos\varphi_1 \cos\varphi_2 \cos\Delta\lambda)$$

$$\Delta\lambda = \lambda_2 - \lambda_1 \ ; \ 0° < \Delta\lambda < 180°$$

$\Delta\lambda$ represents the difference between the longitudes of departure and arrival positions.

Orthodromic distance ($D_O$), expressed in degrees, is then divided into orthodromic unit distances of 0,5º from the point of departure to the point of arrival.

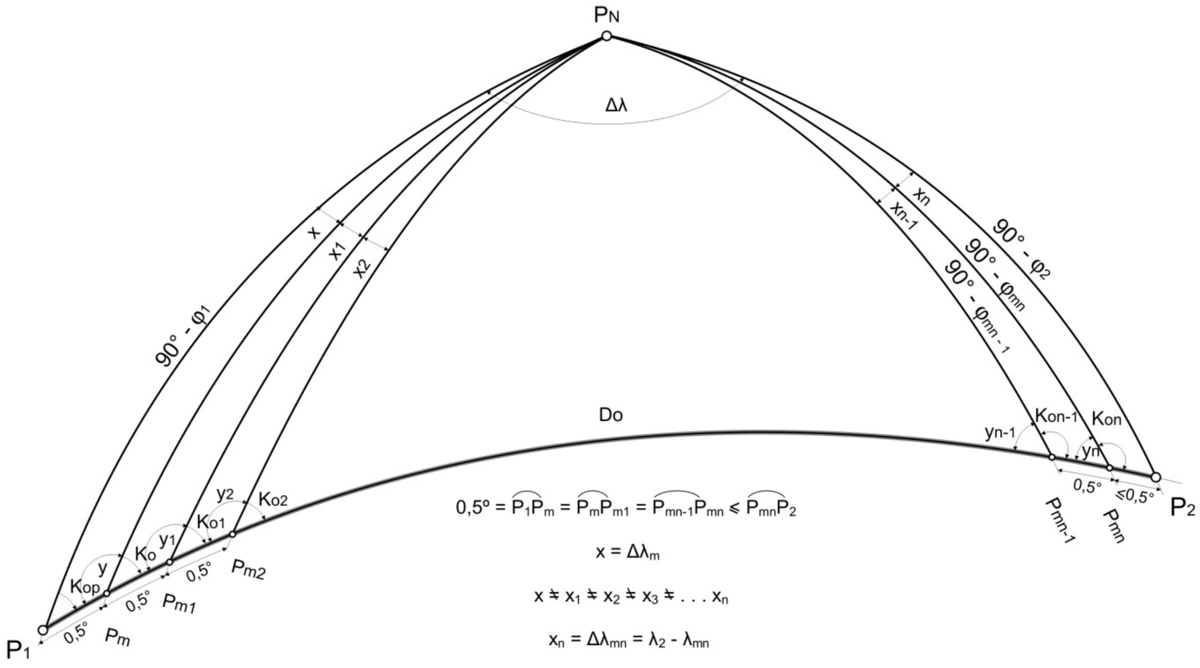


Figure 5: Division of the orthodrome in unit distance intervals of 0,5º ≅ 30 M – The second secant method
Source: Made by authors

### 3.2.1 *Orthodromic interposition coordinates determination*

Applying spherical trigonometry, the interposition coordinates can be determined in different ways. The following equation can be derived from the nautical – positioning spherical triangle:

$$K' = \cos^{-1}\left(\frac{\sin\varphi_2 - \sin\varphi_1 \cos D_O}{\cos\varphi_1 \sin D_O}\right)$$

The Initial Orthodromic Course ($K_{OP}$) is determined by the spherical angle K' with the following relations:

$K_{OP} = K'$ if $\Delta\lambda > 0$

$K_{OP} = 360^\circ - K'$ if $\Delta\lambda < 0$

The following relations can be derived from the spherical triangle $P_1P_NP_2$ shown in Figure 5:

$$\varphi_m = \sin^{-1}\left(\cos 0{,}5^\circ \sin\varphi_1 + \sin 0{,}5^\circ \cos\varphi_1 \cos K_{OP}\right)$$

$$x = \cos^{-1}\left(\frac{\cos 0{,}5^\circ - \sin\varphi_1 \sin\varphi_m}{\cos\varphi_1 \cos\varphi_m}\right)$$

where:

$x = \Delta\lambda_m$ – angle in terrestrial Pole enclosed by the meridians of two adjacent interpositions

$\Delta\lambda_m > 0$ – eastward navigation (E)

$\Delta\lambda_m < 0$ – westward navigation (W)

Interposition coordinates $P_m$ ($\varphi_m$, $\lambda_m = \lambda_1 + x$)

$$y = \cos^{-1}\left(\frac{\sin\varphi_1 - \cos 0{,}5^\circ \sin\varphi_m}{\sin 0{,}5^\circ \cos\varphi_m}\right)$$

y – spherical triangle in the respective orthodromic interposition

$$\begin{Bmatrix} K_0 = 180^\circ \pm y \\ K_0 = 360^\circ - y \end{Bmatrix}$$

$K_0$ – orthodromic course in interposition $P_m$, which depends on the hemisphere on which the ship is sailing (N or S) and sailing direction (E or W)

The coordinates of other orthodromic interpositions from $P_{m1}$ to $P_{mn}$ can be determined with the following equations:

$$\varphi_{mn} = \sin^{-1}\left(\cos 0{,}5^\circ \sin\varphi_{mn-1} + \sin 0{,}5^\circ \sin\varphi_{mn-1} \cos K_{on-1}\right)$$

$$x_n = \Delta\lambda_{mn} = \cos^{-1}\left(\frac{\cos 0{,}5^\circ - \sin\varphi_{mn-1} \sin\varphi_{mn}}{\cos\varphi_{mn-1} - \cos\varphi_{mn}}\right)$$

$$P_{mn}\left(\varphi_{mn}, \lambda_{mn} = \lambda_{mn-1} + x_n\right)$$

$$y_n = \cos^{-1}\left(\frac{\sin\varphi_{mn-1} - \cos 0{,}5^\circ \sin\varphi_{mn}}{\sin 0{,}5^\circ \cos\varphi_{mn}}\right)$$

$$\begin{Bmatrix} K_{on} = 180^\circ \pm y_n \\ K_{on} = 360^\circ - y_n \end{Bmatrix}$$

$K_{on}$ – orthodromic course in interposition $P_{mn}$

### 3.2.2 *Loxodromic course determination*

From one interposition to another, the ship sails in unaltered loxodromic course ($K_L$), calculated by the equation derived from the III. Loxodromic Triangle (the Course Triangle) [Kos, 1996]:

$$\mathrm{tg}K = \frac{\Delta\lambda_m}{\Delta\varphi_{Mm}}$$

where:

$\Delta\lambda_m = \lambda_{mn} - \lambda_{mn-1}$ – longitude difference between two adjacent orthodromic interpositions, expressed in angular minutes

$\Delta\varphi_{Mm} = \varphi_{Mmn} - \varphi_{Mmn-1}$ – Mercator latitudes difference between two adjacent orthodromic interpositions, expressed in angular minutes

If the shape of the Earth is approximated by the shape of the sphere, then:

$$\varphi_{Mm} = 7915{,}7044667898\log\left[\mathrm{tg}\left(45° + \frac{\varphi_{mn}}{2}\right)\right]...[']$$

If the shape of the Earth is approximated by the shape of the biaxial rotation ellipsoid, then [Benković et al, 1986]:

$$\varphi_{Mm} = 7915{,}7044667898\log\left[\mathrm{tg}\left(45° + \frac{\varphi_{mn}}{2}\right)\left(\frac{1 - e\sin\varphi_{mn}}{1 + e\sin\varphi_{mn}}\right)^{\frac{e}{2}}\right]...[']$$

where:

e – the first numerical eccentricity of the ellipsoid

K – the angle in III. loxodromic triangle

$K_L$ – general loxodromic navigation course

The following quadrant navigation cases are possible, which then define loxodromic courses (Figure 6) [Wippern, 1982]:

1 I. navigation quadrant; $\Delta\lambda_m > 0$, $\Delta\varphi_{Mm} > 0$, $K_L = 360° + K = K$
2 II. navigation quadrant; $\Delta\lambda_m > 0$, $\Delta\varphi_{Mm} < 0$, $K_L = 180° + K$
3 III. navigation quadrant; $\Delta\lambda_m < 0$, $\Delta\varphi_{Mm} < 0$, $K_L = 180° + K$
4 IV. navigation quadrant; $\Delta\lambda_m < 0$, $\Delta\varphi_{Mm} > 0$, $K_L = 360° + K$

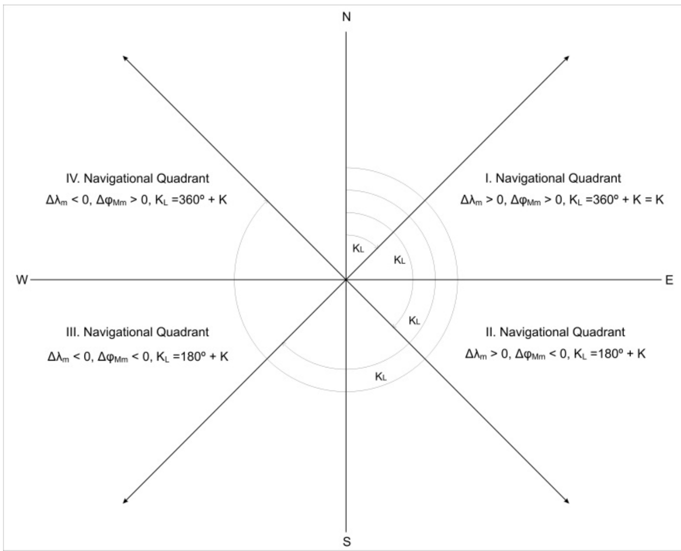


Figure 6: Four possibilities of Quadrant Navigation

Source: Made by authors

From the initial position $P_1$ to the interposition $P_m$ the ship sails in loxodromic course $K_L$. Then, between $P_m$ and $P_{m1}$ in course $K_{L1}$, between $P_{m1}$ and $P_{m2}$ in $K_{L2}$, ... from the interposition $P_{mn-1}$ to $P_{mn}$ the ship sails in loxodromic course $K_{Ln}$, and finally, from $P_{mn}$ to the arrival position $P_2$ in the last loxodromic course. The length of the last stage of navigation between $P_{mn}$ and $P_2$ is always ≤ 30 M.

If, while navigating, the ship is not placed on the planned orthodromic path[3], new orthodrome is calculated from exact current position towards the position of arrival, and the procedure is then repeated, dividing the new orthodrome in unit distance intervals of 0,5º, and calculating navigation elements again [Kos, 1996].

## 4 APPROXIMATION OF ORTHODROMIC NAVIGATION BY THE TANGENT METHOD – ORTHODROME DIVISION IN UNIT COURSE ALTERATIONS

Instead of secants determined by the interpositions (the first secant method), or the unit distance intervals (the second secant method), the navigation is here approximated by the tangent lines of the orthodrome, i.e. unit orthodromic course alterations (ΔK) are derived as follows [Zorović et al, 2010]:

- the Initial Orthodromic Course in position $P_1$ ($K_{OP}$) and the Final Orthodromic Course ($K_{OK}$) in position $P_2$ are calculated. The following values are then calculated:

[3] For example, by the ship's drift due to the sea currents, the wind, waves, the collision avoidance, etc.

$$x = \frac{(K_{OK} - K_{OP})}{\Delta K}$$

$$D_X = \frac{D_O}{x}[M]$$

$$\text{for } \Delta K = 1° \rightarrow D_X = \frac{D_O}{(K_{OK} - K_{OP})}$$

where:

$x$ – total amount of orthodromic course alteration

$\Delta K$ – 1º, 2º, 3º... arbitrarily selected orthodromic unit course alteration value

$D_O$ – orthodromic distance between positions $P_1$ and $P_2$

$K_{OP}$ – the initial orthodromic course in departure position $P_1$

$K_{OK}$ – the final orthodromic course in arrival position $P_2$

$D_X$ – unit orthodromic distance

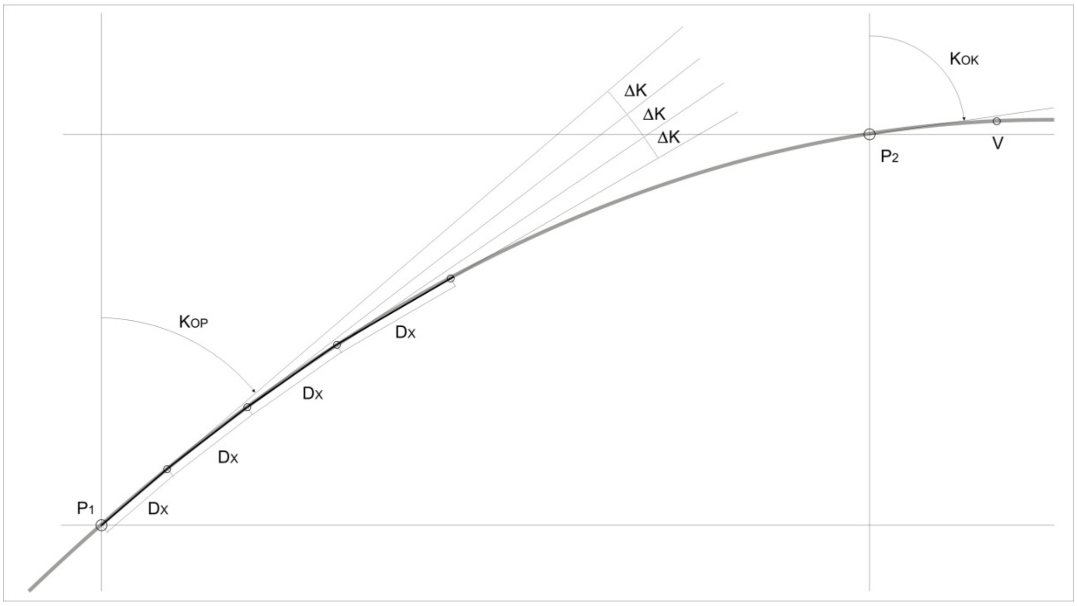


Figure 7: Approximation of orthodromic navigation by the tangent method
Source: Made by authors

## 5 CONCLUSION

From a theoretical point of view, it is not possible to navigate on great circle. The orthodrome navigation is the shortest, while the loxodrome is acceptable nautically, given that the navigation here is obtained in constant, general loxodromic course (with the longer distance travelled) [Bowditch, 1984]. Using the combination of this navigation curves, the problem is solved in a way that the features of the orthodrome as a shortest distance between two points on Earth are maximally utilized. The proposed goal of navigation is thus fulfilled from the practical point of view, given that the great circle is divided in unit values of the specific elements, depending of the method used, on which the navigation is then carried out in loxodromic courses, and loxodromic distances respectively.

Three approximation models of orthodromic navigation have been elaborated in the paper. The first model determines the orthodromic interpositions, which can be calculated from the orthodrome vertex or the initial position, depending on the position of the Vertex, whether it is placed inside the position of departure and arrival or not. With this method, the ship sails in unequal distance intervals. The second model implies the division of the orthodrome in unit distance intervals with the amount of 0,5º $\cong$ 30 M. Hereby, the orthodromic interposition coordinates are determined, and the ship sails in constant loxodromic courses between them. This method is the most accurate of all of the three elaborated. In the third method, the orthodromic navigation is approximated by the determination of the orthodromic unit course alterations $\Delta K$. Here, it is first necessary to calculate the unit course alterations, after which unit orthodromic distances are defined, expressed in nautical miles, representing the navigation of the vessel in the specific course, in a way that the required alteration $\Delta K$ would appear.

The extent to which the navigation will be orthodrome – like, depends on several parametres – considering a specific navigation case, and taking navigation courses and distances between two positions into account. In the Equator and Meridian sailing, the orthodrome and the loxodrome overlap – their distances are equal. This also applies to smaller distances between positions, where there are no discrepancies between these curves. However, in certain cases, the difference between these two distances reaches noticeable values, and then, by approximating the orthodrome, the time spent in navigation can be significantly reduced.


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