TRANSNAV
http://www.transnav.eu
the International Journal
on Marine Navigation
and Safety of Sea Transportation
Volume 18
Number 2
June 2024

DOI: 10.12716/1001.18.02.17

A New Result on Description of the Spectrum of a Sampled Signal

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ABSTRACT: It is explained, in introduction of this paper, why the description of the output signal at an A/D converter in the form that is presented in such respected textbooks as: a one written by Prandoni and Vetterli, and another one by van de Plassche is appropriate and correct. Unlike all others, especially those using in it the so-called comb of Dirac deltas. The latter ones do not lead to getting a correct formula for the spectrum of the output waveform of an A/D converter, or they yield no formula at all. Using the description of the A/D output signal in form of a step function (as in the textbooks mentioned above), a new, correct formula for calculating the spectrum of the sampled signal is derived in this paper. It is a revised version of the formula currently used in the literature, that is of the so-called Discrete-Time Fourier Transform (DTFT), and it is a product of this DTFT and a certain correction factor. Finally, some literature items are referred to in which the designers of integrated circuits (containing A/D converters) point out discrepancies that arise in designs when the multiplying factor mentioned above is not taken into account.

1 INTRODUCTION

Using a scheme similar to that one which is shown in Fig. 1, Prandoni and Vetterli in their book (Prandoni P. & Vetterli M. 2008; on page 284) explained the behavior and operation of an A/D converter.

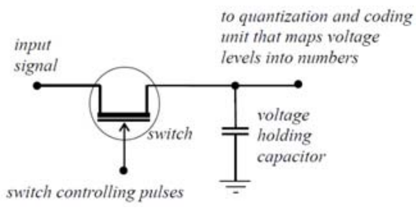

Figure 1. A basic idea of an A/D converter illustrated with the use of a FET transistor switch connected to a capacitor holding the value of an analog input signal taken at the time of switching.

A similar scheme as that shown in Fig. 1 (in a block form) can be also found in a well-known monograph of van de Plassche (van de Plassche R. 1994; on page 4). (That is in a book specially dedicated to the theory of A/D and D/A conversions and to the discussion of integrated circuits that perform these operations.)

The block labeled S/H in Fig. 2 implements the so-called operation of taking a sample value at a time instant kT, $k = \ldots,-2,-1,0,1,2,\ldots$ and maintaining it for a time period T. Such an operation is performed by that electronic circuit, which is shown in Fig. 1, and which consists of a FET transistor and a capacitor connected to its “source” terminal. The FET transistor, controlled by pulses applied to its “gate” terminal, opens regularly every T seconds and charges the capacitor to a current value of the voltage at its "source" terminal. The task of capacitor is to sustain this voltage value for a time equal at least to T. The voltage values, which change in steps on the capacitor, are subjected to a quantization process, and further these quantized values are mapped into numbers.

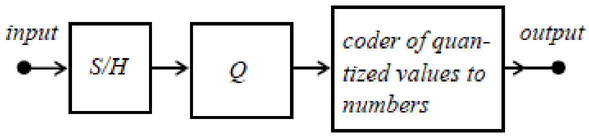

Figure 2. A block diagram of an A/D converter consisting of three blocks: a sample and hold (S/H) unit, an amplitude quantization unit (Q), and a mapper (coder) of quantized values to numbers .

It follows from Figs. 1 and 2, and descriptions of the equivalent circuits of an A/D converter shown in these figures that its output waveforms (understood as functions of a continuous time *t*) have the form of a slightly disturbed step function presented in Fig. 3. The shape of actual waveforms at the outputs of A/D converters is more rich than the step function shown in Fig. 3 and depends upon the architecture and technology in which a given converter is implemented. This shape is characterized by such parameters as: settling time, acquisition time, aperture, aperture jitter, hold mode settling time, hold mode feedthrough, droop; see, for example, (van de Plassche R. 1994; page 74). However, from the point of view of a designer of signal processing systems, most of these parameters are of secondary importance (which does not mean at all that they are not relevant to designers of their integrated structures in specific semiconductor technologies). In the description visualized in Fig. 3, we restricted ourselves to pointing out that in each time interval *<kT, (k+1)T>*, *k = …,-2,-1,0,1,2,…* we have its initial segment (the so-called track part and beginning of the hold part) rich in changes and the second one (covering the almost entire hold part) already stabilized on the hold voltage level of a given time interval. The former segment is denoted on the waveform of Fig. 3 by a diagonal dash, while the latter is marked with a longer dash, parallel to the time axis – at each of the aforementioned time intervals.

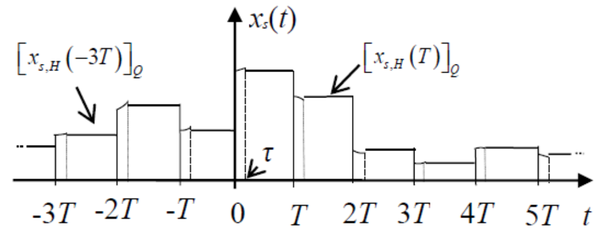

Figure 3. Sketch illustrating the form of the waveform of an example sampled signal, denoted by $x_s(t)$ and related with an un-sampled one *x(t)* (not shown here). (This figure is based on a one, which was used in discussions presented in (Borys A. 2022)).

In Fig. 3, $[x_{s,H}(-3T)]_Q$ and $[x_{s,H}(T)]_Q$, where the lower index *Q* means the operation of amplitude quantization, stand for illustration of the quantized values of the sampled signal $x_s(t)$. These values are assigned to the following instants: *-3T* and *T*, respectively, and worked out in the hold parts of the corresponding time intervals (mentioned above). In Fig. 3, it is assumed that the track part (including beginning of the hold part, too) lasts τ seconds, and the track and hold parts together last *T* seconds.

As we know from the literature, the idealized version of the signal sampling process neglects the switching time in this process, assuming that the time τ is much smaller than *T*. In other words, in an ideal case, τ = 0 is assumed. Then the waveform shown in Fig. 3 takes the form which is visualized in Fig. 4.

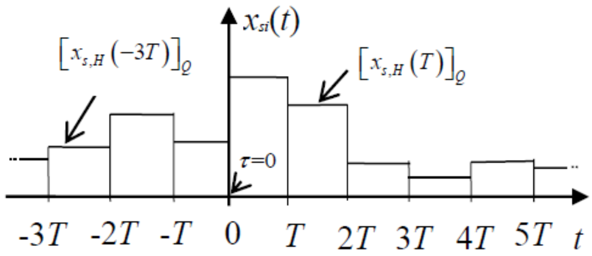

Figure 4. Sketch illustrating an idealized version of the waveform shown in Fig. 3; it is denoted here by $x_{si}(t)$. (This figure is based on a one, which was used in discussions presented in (Borys A. 2022)).

Furthermore, note that the coder of quantized values shown in Fig. 2 plays, in addition to performing the conversion of these values into numbers, a role of an element that holds a number it generated at a given instant, as an encoder, for exactly *T* seconds – before feeding it further into a signal processor or a signal processor buffer. Therefore, the waveform at the output of the decoder is exactly the same as the one shown in Fig. 4 (in this idealized version), except that the quantized values are now "scaled" to numbers. For completeness of the picture of what appears as the final result at the output of an A/D converter, the waveform in Fig. 4 is redrawn to a “scaled” one shown in Fig. 5.

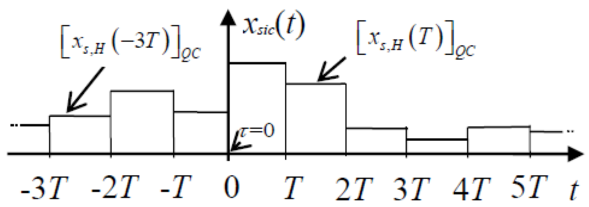

Figure 5. Sketch illustrating an idealized version of the waveform shown in Fig. 3 after performing amplitude quantization and coding into numbers; it is denoted here by $x_{sic}(t)$. (This figure is based on a one, which was used in discussions presented in (Borys A. 2022)).

In Fig. 5, $[x_{s,H}(-3T)]_{QC}$ and $[x_{s,H}(T)]_{QC}$, where the lower index QC means performing both the operations: amplitude quantization and coding (one after the other), are example values of the quantized and coded signal $x_{sic}(t)$.

In order to make further comparisons of the ideal descriptions of the sampled signal presented above, let us also add to them the one describing the sampled signal immediately before performing the amplitude quantization on it. But we give up here a graphical illustration of it since a waveform in the figure would have the same form as that one shown in Fig. 4, with the only difference in that the values of the "stair steps" on it would differ slightly from the corresponding ones in Fig. 4. Furthermore, these values would belong to the set of real numbers. Finally, let us call this waveform as $x_{si\bar{q}}(t)$.

Now, using the above signal and notation, we see that the signal determining the quantization error $e_q(t)$ can be expressed according to the following equation (Oppenheim A. V., Schafer R. W., Buck J. R. 1998):

$$x_{si}(t) = x_{si\bar{q}}(t) + e_q(t) . \tag{1}$$

That is we get from (1a)

$$e_q(t) = x_{si}(t) - x_{si\bar{q}}(t) . \tag{2}$$

Note that in our considerations presented here the signal $x_{si\bar{q}}(t)$ could be also interpreted otherwise. For example, as a one with the values averaged in each of the time intervals *<kT, (k+1)T>, k = …,-2,-1,0,1,2,…* – over a part or in the whole time interval (of the length *T*). Such a proposal was made, for instance, by Vetterli M., Kovacevic J., and Goyal V. K. in their book (Vetterli M., Kovacevic J., Goyal V. K. 2014; on page 45). Further, note that there are also other possible interpretations of the signal denoted here as $x_{si\bar{q}}(t)$.

But we see here clearly that all of them differ or would differ, more or less, in values of the voltage levels on the corresponding "stair steps" of a step function that describes their shape. That is like such a function as the one visualized in Fig. 4 but before quantization operation. So we can characterize all of them through the following equation:

$$e_s(t) = x_{si\bar{q}}(t) - x_{si\bar{q}}(\lfloor t/T \rfloor T) , \tag{3}$$

where $e_s(t)$ stands for an error related with the sampling process (as it has been depicted here). And $\lfloor \cdot \rfloor$ means the floor function in (3).

Eq. (3) can be rewritten in the form

$$x_{si\bar{q}}(t) = x_{si\bar{q}}(\lfloor t/T \rfloor T) + e_s(t) , \tag{4}$$

which allows us to say that all the descriptions of the sampled operation immediately before performing the amplitude quantization, mentioned just above, differ from each other only in the error function $e_s(t)$. (Because all the values $x_{si\bar{q}}(\lfloor t/T \rfloor T)$ for these functions are the same.)

Introducing (4) into (1) gives

$$x_{si}(t) = x_{si\bar{q}}(\lfloor t/T \rfloor T) + e_s(t) + e_q(t) . \tag{5}$$

And this result shows us that the sampled signal, before performing its coding, is subject to two errors: $e_s(t)$ and $e_q(t)$ both related with the uncertainty in amplitude.

Obviously, in the process of sampling a signal, we have also to do with uncertainty in determining the sampling time instants. This, however, has no significant impact on the shape of the $x_s(t)$ waveforms. At most, they are shifted on the time axis by a constant value, and with a small time jitter the changes of widths of "stair steps" of a step function (as, for example, of the one in Fig. 4) can be neglected.

2 CALCULATION OF THE SAMPLED SIGNAL SPECTRUM

Discussion of the cases presented in the previous section showed that the most appropriate description of the waveform at the output of an A/D converter is a step function – independently whether we understand by it the signal immediately before performing the quantization operation or the already quantized one, or the coded latter one (locally, at each of the "stair steps" of this function). And, as already recognized, the cases mentioned above differ from each other only by small deviations in the values of the voltage levels of the corresponding "stair steps" of their step functions. However, this is irrelevant to the problem considered in this paper. What is important here is the staircase character of the function, which describes what happens at the output of an A/D converter. And this allows us to describe all of these cases analytically through a single (generic) formula, as follows:

$$x_g(t) = x_g(kT) \ \text{ for } \ kT \le t < (k+1)T \quad \text{and with} \ \ k = \lfloor t/T \rfloor , \tag{6}$$

where $x_g(t)$ stands for the value of the *k*-th "stair step" of the step function $x_{si\bar{q}}(t)$ or $x_{si}(t)$, or also $x_{sic}(t)$. Moreover, for the sake of clarity, it is worth recalling at this point that the error functions $e_s(t)$ and $e_q(t)$ are, obviously, also step functions.

Let us now calculate the Fourier transform of the waveform given by (6). We will do this in detail starting with

$$\begin{aligned} X_g(f) &= \mathcal{F}(x_g(t)) = \int_{-\infty}^{\infty} x_g(t)\exp(-j2\pi ft)dt = \\ &= \ldots + \int_{-T}^{0} x_g(-T)\exp(-j2\pi ft)dt + \int_{0}^{T} x_g(0) \cdot \\ &\cdot \exp(-j2\pi ft)dt + \\ &+ \int_{T}^{2T} x_g(T)\exp(-j2\pi ft)dt + \ldots \end{aligned} , \tag{7}$$

where $X_g(t) = \mathcal{F}(x_g(t))$ means the Fourier transform of the signal $x_g(t)$, f is the frequency, and $j = \sqrt{-1}$.

In the next step, calculating integrals in (7), we get

$$X_g(f) = ... + \frac{x_g(-T)}{-j2\pi f}\exp(-j2\pi ft)\Big|_{-T}^{0} +$$
$$+\frac{x_g(0T)}{-j2\pi f}\exp(-j2\pi ft)\Big|_{0}^{T} + \frac{x_g(T)}{-j2\pi f}\cdot$$
$$\cdot\exp(-j2\pi ft)\Big|_{T}^{2T} + ... =$$
$$= \frac{j}{2\pi f}\{... - x_g(-T)\exp(j2\pi fT) +$$
$$+ x_g(-T)\exp(-j2\pi f(0T)) - x_g(0T)\cdot$$
$$\cdot\exp(-j2\pi f(0T)) + x_g(0T)\exp(-j2\pi fT) -$$
$$- x_g(T)\exp(-j2\pi fT) + x_g(T)\cdot$$
$$\cdot\exp(-j2\pi f(2T)) + ...\} = \frac{j}{2\pi f}\{... + [x_g(-2T) -$$
$$- x_g(-T)]\exp(j2\pi fT) + [x_g(-T) - x_g(0T)]\cdot$$
$$\cdot\exp(j2\pi f(0T)) + [x_g(0T) - x_g(T)]\cdot$$
$$\cdot\exp(-j2\pi fT) + [x_g(T) - x_g(2T)]\cdot$$
$$\cdot\exp(-j2\pi f(2T)) + ...\} \quad . \tag{8}$$

In a compact form, (8) can be expressed as

$$X_g(f) = \frac{1}{j2\pi f}\sum_{k=-\infty}^{\infty}[x_g(kT) - x_g((k-1)T)]\cdot$$
$$\cdot\exp(-j2\pi fkT) \overset{df}{=} \frac{1}{j2\pi f}\mathrm{DTFT}([x_g(kT) -$$
$$- x_g((k-1)T)]) \quad , \tag{9}$$

where $\mathrm{DTFT}(\cdot)$ means the so-called Discrete Time Fourier Transform (Oppenheim A. V., Schafer R. W., Buck J. R. 1998), (Vetterli M., Kovacevic J., Goyal V. K. 2014). (Moreover, note that (9) includes, at the same time, also the definition of this transform.)

Further, observe that (9) can rewritten as

$$X_g(f) = \frac{1}{j2\pi f}\{\mathrm{DTFT}(x_g(kT)) -$$
$$- \mathrm{DTFT}(x_g((k-1)T))\} \quad . \tag{10}$$

Moreover, we can transform $\mathrm{DTFT}(x_g((k-1)T))$ occurring in (10) to the following form:

$$\mathrm{DTFT}(x_g((k-1)T)) = \sum_{k=-\infty}^{\infty} x_g((k-1)T)\cdot$$
$$\cdot\exp(-j2\pi fkT) = \sum_{k'=-\infty}^{\infty} x_g(k'T)\cdot$$
$$\cdot\exp(-j2\pi f(k'+1)T) = \exp(-j2\pi fT)\cdot \tag{11}$$
$$\cdot\sum_{k'=-\infty}^{\infty} x_g(k'T)\exp(-j2\pi fk'T) =$$
$$= \mathrm{DTFT}(x_g(kT))\exp(-j2\pi fT) \quad .$$

Note that to get a final result in (11) we introduced first an auxiliary variable $k' = k - 1$ there, then performed some manipulations, and finally dropped prime symbol at k'. Substituting next this result to (10), we get

$$X_g(f) = \frac{\mathrm{DTFT}(x_g(kT))}{j2\pi f}(1 - \exp(-j2\pi fT)) =$$
$$= T\cdot\mathrm{DTFT}(x_g(kT))\cdot$$
$$\cdot\frac{(\exp(j\pi fT) - \exp(-j\pi fT))}{j2\pi fT\exp(j\pi fT)} = \tag{12}$$
$$= T\cdot\mathrm{DTFT}(x_g(kT))\exp(-j\pi fT)\frac{\sin(\pi fT)}{\pi fT} =$$
$$= T\cdot\mathrm{DTFT}(x_g(kT))\mathrm{sinc}(\pi fT)\exp(-j\pi fT) \quad .$$

In this way, we arrived at the end of our calculations of the spectrum of a sampled signal. Hence, our final result that follows from (14) has the following form:

$$X_g(f) =$$
$$= T\cdot\mathrm{DTFT}(x_g(kT))\mathrm{sinc}(\pi fT)\exp(-j\pi fT) \,. \tag{13}$$

3 DISCUSSION OF THE RESULT ACHIEVED

Evidently, the result given by (13) differs from the one telling us that the DTFT of a sequence of analog signal samples is a Fourier transform of this sequence. In other words, the DTFT is identified with (or plays a role of) a Fourier transform (spectrum) of the sampled version of an analog signal, and it turns out here that this is not a quite correct identification. Furthermore, this identification is done without any justification, simply assuming a priori it as a definition.

Moreover, as shown in Introduction, the above approach is difficult to justify when confronted with signals that appear at the output of an A/D converter. Undoubtedly, these are continuous-time signals so calculation of their Fourier transforms (spectra) can be done within the framework of a classical Fourier theory – without any problems. It is not necessary to reach for some new tools, such as, for example, DTFT. The result obtained in this paper shows that the above is possible. Besides, our tackling with the problem is firmly "anchored" in realities of physical operations of sampling analog signals. Additionally, it also turns out that the outcomes can be expressed in terms of the DTFT and a certain multiplying function, see (13).

Note, however, that for very small values of the frequency *f*, compared to the value of the sampling frequency $f_s=1/T$, i.e. for $f/f_s \ll 1 \Rightarrow f \ll f_s$, it can be assumed that the values of the multiplying function in (13) are approximately equal to 1. And this allows us, for these frequency ranges, to assume that the following formula:

$$X_g(f) \cong T\cdot\mathrm{DTFT}(x_g(kT)) \tag{14}$$

is valid approximately.

By the way, note the occurrence of a scaling factor T in equations (12), (13), and (14) above. Its existence is due to the fact that the DTFT is only a weighted sum of signal samples, while a "real" signal spectrum is related to integration over time, or "summation times time."

Obviously, the expression given by (13) for calculation of the spectrum of a sampled signal differs also from the following highly celebrated formula:

$$X_{sd}\left(f \right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(f - kf_s \right), \tag{15}$$

where $X(f)$ means the Fourier transform of an un-sampled signal $x(t)$, but $X_{sd}(f)$ is identified with the spectrum of its sampled version $x_{sd}(f)$ (described in another way as $x_g(t)$ here). To see this, let us use in (15) the relationship (Oppenheim A. V., Schafer R. W., Buck J. R. 1998), (Vetterli M., Kovacevic J., Goyal V. K. 2014) according to which the DTFT equals the expression occurring on the right-hand side of (15). Evidently, we see then that $X_g\left(f \right) \neq T \cdot X_{sd}\left(f \right)$ holds. Moreover, we get then the equivalent of (13) in the following form:

$$X_g\left(f \right) = \operatorname{sinc}\left(\pi fT \right)\exp\left(-j\pi fT \right) \cdot \\ \cdot \sum_{k=-\infty}^{\infty} X\left(f - kf_s \right) \quad . \tag{16}$$

So we can say that the formulas (13) and (16) postulate the need to introduce a correcting coefficient (correcting function): $\operatorname{sinc}\left(\pi fT \right)\exp\left(-j\pi fT \right)$ into the expression on the right-hand side of (15), which is currently in force in the literature.

4 CONCLUSIONS

The need to develop a new, better description of output waveforms of A/D converters have been indirectly confirmed in many places in the literature. For example, see (de la Rosa J., Perez-Verdu B., Medeiro F., del Rio R., Rodriguez-Vazquez A. 2001) and (de la Rosa J., Perez-Verdu B., Medeiro F., del Rio R., Rodriguez-Vazquez A. 2004). Among others, these authors confirm that the effect which is discussed in this paper is most annoyed at the large ratios of signal frequencies to sampling frequencies (comparable to each other). And this effect actually disappears at very high sampling frequencies (that is, when the above frequency ratio is relatively small; see (14)).

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