Positive Descriptor Time-varying Discrete-time Linear Systems and Their Asymptotic Stability

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ABSTRACT: The positivity and asymptotic stability of the descriptor time-varying discrete-time linear systems are addressed. The Weierstrass-Kronecker theorem on the decomposition of the regular pencil is extended to the time-varying discrete-time descriptor linear systems. Using the extension necessary and sufficient conditions for the positivity of the systems are established. Sufficient conditions for asymptotic stability of the positive systems are presented. The effectiveness of the tests is demonstrated on the example.

1 INTRODUCTION

A dynamical system is called positive if its trajectory starting from any nonnegative initial condition state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive system theory is given in the monographs Farina & Rinaldi 2000, Kaczorek 2001 and in the papers Kaczorek 1997, 1998a, 2011, 2015. Models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc.

The Lyapunov, Bohl and Perron exponents and stability of time-varying discrete-time linear systems have been investigated in Czornik et. all 2012, 2013a, 2013b, 2013c, 2013d, 2014. The positive standard and descriptor systems and their stability have been analyzed in Kaczorek 1998a, 2001, 2011, 2015. The positive linear systems with different fractional orders have been addressed in Kaczorek 2011, 2012 and the singular discrete-time linear systems in Kaczorek 1998a. The switched discrete-time systems have been considered in Zhang et. all 2014a, 2014b and the extremal norms for positive linear inclusions in Zhong et. all 2013.

In this paper the positivity and asymptotic stability of the descriptor time-varying discrete-time linear systems with regular pencils will be investigated.

The paper is organized as follows. In section 2 the Weierstrass-Kronecker decomposition of the regular pencil is extended to descriptor time-varying discrete-time linear systems and the solution of the state-equation describing the time-varying discrete-time linear system is derived. Necessary and sufficient conditions for the positivity of the descriptor systems are established in section 3. The stability of the positive descriptor systems is addressed in section 4. Concluding remarks are given in section 5.

The following notation will be used: $\mathbb{R}$ - the set of real numbers, $\mathbb{R}_{n \times m}^{+\infty}$ - the set of $n \times m$ real matrices, $\mathbb{R}_{n \times m}^{+\infty}$ - the set of $n \times m$ matrices with nonnegative entries and $\mathbb{R}_+ = \mathbb{R}_+^{+\infty}$, $I_n$ - the $n \times n$ identity matrix.
2 POSITIVE TIME-VARYING DISCRETE-TIME LINEAR SYSTEMS

Consider the descriptor time-varying discrete-time linear system

\[ E(i)x_{i+1} = A(i)x_i + B(i)u_i, \quad i \in Z_+ = \{0,1,\ldots\} \quad (2.1a) \]

\[ y_i = C(i)x_i \quad (2.1b) \]

where \( x_i \in \mathbb{R}^n \), \( u_i \in \mathbb{R}^m \), \( y_i \in \mathbb{R}^p \) are the state, input and output vectors and \( A(i) \in \mathbb{R}^{nxn}, \ B(i) \in \mathbb{R}^{nxm}, \ C(i) \in \mathbb{R}^{pxn} \) are matrices with entries depending on \( i \in Z_+ \).

It is assumed that \( \det E(i) = 0 \), \( i \in Z_+ \) and

\[ \det[E(i)\lambda - A(i)] \neq 0 \quad (2.2) \]

for some \( \lambda \in \mathbb{C} \) (the field of complex numbers) and \( i \in Z_+ \).

It is well-known (Kaczorek 2015) that if (2.2) holds then there exists a pair of nonsingular matrices \( P(i), Q(i) \in \mathbb{R}^{nxn} \) such that

\[ P(i)[E(i)\lambda - A(i)]Q(i) = \begin{bmatrix} I_n & 0 \\ 0 & N \end{bmatrix} \lambda - A(i), \quad (2.3) \]

where \( i \in Z_+ \), \( n_i = \text{deg} [\det (E(i)\lambda - A(i))] \), \( A(i) \in \mathbb{R}^{nxn} \), \( N \in \mathbb{R}^{n_i \times n_i} \) is the nilpotent matrix with the index \( \mu \) (i.e. \( N^\mu = 0 \) and \( N^{\mu+1} \neq 0 \)).

The matrices \( P(i), Q(i), A(i) \) can be found by for example the use of elementary row and column operations (Kaczorek 1998b).

Premultiplying (2.1a) by the matrix \( P(i) \), introducing the new state vector

\[ \bar{x}_i = Q^{-1}(i)x_i = \begin{bmatrix} x_{i,1} \\ x_{2i,1} \\ \vdots \\ x_{ni,i} \\ x_{2i,2} \\ \vdots \\ x_{2ni,i} \end{bmatrix}, \quad \bar{y}_i = \begin{bmatrix} \bar{x}_{1,1} \\ \bar{x}_{2,1} \\ \vdots \\ \bar{x}_{ni,1} \\ \bar{x}_{2,2} \\ \vdots \\ \bar{x}_{2ni,2} \end{bmatrix} = \begin{bmatrix} x_{1,1} \\ x_{2,1} \\ \vdots \\ x_{ni,1} \\ x_{2,2} \\ \vdots \\ x_{2ni,2} \end{bmatrix} = \begin{bmatrix} \bar{x}_{1,1} \\ \bar{x}_{2,1} \\ \vdots \\ \bar{x}_{ni,1} \\ \bar{x}_{2,2} \\ \vdots \\ \bar{x}_{2ni,2} \end{bmatrix} = \begin{bmatrix} x_{1,1} \\ x_{2,1} \\ \vdots \\ x_{ni,1} \\ x_{2,2} \\ \vdots \\ x_{2ni,2} \end{bmatrix} \quad (2.4) \]

and using (2.3) we obtain

\[ \bar{x}_{i+1} = A(i)\bar{x}_i + B(i)u_i \quad (2.5a) \]

\[ N\bar{x}_{2i+1} = \bar{x}_{2i} + B(i)u_i \quad (2.5b) \]

where

\[ P(i)B(i) = \begin{bmatrix} B_1(i) \\ B_2(i) \end{bmatrix}, \quad B_1(i) \in \mathbb{R}^{nxm}, \ B_2(i) \in \mathbb{R}^{nxm}. \quad (2.5c) \]

Theorem 2.1. The solution of equation (2.5a) for known initial condition \( \bar{x}_{10} \in \mathbb{R}^n \) and input \( u_i \in \mathbb{R}^m \), \( i \in Z_+ \) is given by

\[ \bar{x}_{i,j} = \Phi_1(i,0)\bar{x}_{i,0} + \sum_{j=0}^{i-1} \Phi_1(i,j+1)B_1(j)u_j, \quad i \in Z_+ \quad (2.6a) \]

where

\[ \Phi_1(k,j) = \begin{cases} I_n & \text{for } k = j \geq 0 \\ A(k-1)A(k-2)\ldots A(j) & \text{for } k > j \geq 0 \end{cases} \quad (2.6b) \]

Proof is given in (Kaczorek 2015).

To simplify the notation it is assumed that the matrix \( N \) in (2.5b) has the form

\[ N = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \quad (2.7) \]

From (2.5b) and (2.7) we have

\[ \begin{bmatrix} \bar{x}_{2,1} \\ \bar{x}_{2,2} \\ \vdots \\ \bar{x}_{2ni,1} \\ \bar{x}_{2ni,2} \end{bmatrix} = \begin{bmatrix} B_1(i) \\ B_2(i) \end{bmatrix} + \begin{bmatrix} x_{2,1} \\ x_{2,2} \\ \vdots \\ x_{2ni,1} \\ x_{2ni,2} \end{bmatrix} \quad (2.8a) \]

for \( i \in Z_+ \) and

\[ 0 = \bar{x}_{2n_i,j} + B_{2n_i}(i)u_i, \]

\[ \bar{x}_{2n_i,j+1} = \bar{x}_{2n_i,j} + B_{2n_i-1}(i)u_i, \quad i \in Z_+. \quad (2.8b) \]

Solving the equations (2.8b) with respect to the components of the vector \( \bar{x}_{2,j} \) we obtain

\[ \bar{x}_{2n_i,j} = -B_{2n_i}(i)u_i, \]

\[ \bar{x}_{2n_i-1,j} = -B_{2n_i}(i+1)u_{i+1} - B_{2n_i-1}(i)u_i, \]

\[ \vdots \]

\[ \bar{x}_{21,j} = -B_{2n_i}(i + n_{i-1} - 1)u_{i+n_{i-1}} - \ldots - B_{21}(i)u_1. \quad (2.9) \]

The considerations can be easily extended to the case when the matrix \( N \) in (2.5b) has the form

\[ N = \text{blockdiag}[N_1,\ldots,N_q], \quad q > 1 \quad (2.10) \]

and \( N_k \) for \( k = 1,2,\ldots,q \) has the form (2.7).

Example 2.1. Consider the descriptor time-varying system described by the equation (2.1a) with the matrices
\[
E(i) = \begin{bmatrix}
0 & 0 & 0 & e^{2i} \\
(i + 2)(\sin(i) + 1) & 0 & 0 & 0 \\
i + 1 & 0 & 0 & 0 \\
i + 2 & 0 & 0 & 0 \\
\end{bmatrix},
\]
\[
P(i) = \begin{bmatrix}
1 + e^{-i} & 1 + \sin(i) & 0 & 0 \\
0 & 0 & 1 & 1 + \cos(i) \\
0 & 0 & 0 & i + 1 \\
0 & 0 & 0 & i + 2 \\
\end{bmatrix},
\]
\[
Q(i) = \begin{bmatrix}
0 & i + 1 & 0 & 0 \\
0 & i + 2 & 0 & 0 \\
-2 + \cos(i) & 0 & 0 & 0 \\
0 & 0 & 0 & e^{-2i} \\
\end{bmatrix},
\]
\[
I_{n_1} = 0,
\]
\[
I_{n_2} = 0,
\]
\[
0 & 0 & 0 & 1
\]
and
\[
I_{n_1} = P(i)E(i)Q(i) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix},
\]
\[
A_i(i) = \begin{bmatrix}
0 & 0 & a_{i3}(i) & 0 \\
a_{i2}(i) & a_{i2}(i) & a_{i3}(i) & a_{i4}(i) \\
a_{i3}(i) & 0 & 0 & a_{i4}(i) \\
0 & 0 & 0 & a_{i4}(i) \\
\end{bmatrix},
\]
\[
B_i(i) = P(i)B(i) = \begin{bmatrix}
e^{-i} & 0 & 0 & 0 \\
0 & e^{-i} & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 2i & 0 \\
\end{bmatrix},
\]
where
\[
a_{i3}(i) = \frac{1}{\cos(i) + 2},
\]
\[
a_{i2}(i) = \frac{(i + 2)(i + 2\cos(i) + 2\sin(i) + i\sin(i) + \cos(i)\sin(i) + 3)}{(i + 1)(\sin(i) + 2)},
\]
\[
a_{i2}(i) = e^{2i} - 2e^i, \quad a_{i3}(i) = -\frac{e^{-1} + 1}{\cos(i) + 2},
\]
\[
a_{i4}(i) = \frac{e^{2i} + 2\cos(i)(\sin(i) + 1)}{i + 2},
\]
\[
a_{i3}(i) = \frac{-i + 2}{\sin(i) + 2}, \quad a_{i4}(i) = -\frac{e^{2i}(i + 2)(\cos(i) + 1)}{i + 1},
\]
\[
a_{i4}(i) = \frac{e^{2i}(i + 2)}{i + 1}.
\]

The condition (2.2) is satisfied since
\[
\det[E(i)\lambda - A(i)] = -(i + 2)^2(2\lambda + \lambda^2 - 1)(2\lambda + i + \lambda\sin(i) + 1) \neq 0
\]
\[
(2.12)
\]

In this case

\[
(2.13)
\]

The condition (2.2) is satisfied since

The equation (2.5) have the form
\[
(\bar{x}_{1,i+1}) = \begin{bmatrix} e^{-i} \\
0 \end{bmatrix} + \frac{1 + \cos(i)}{i + 1 - 2\cos(i)} \times \begin{bmatrix} x_{1,i} \\
0 \end{bmatrix}
\]
\[
(2.15a)
\]

and
\[
(2.15b)
\]

The solution of (2.15a) is given by (2.6) with the matrices \( A_i(i) \) and \( B_i(i) \) defined by (2.14).

From (2.15b) we have
\[
\bar{x}_{2,j} = -2iu_{2j},
\]
\[
(2.16)
\]

The solution of the equation (2.1a) with (2.11) is given by

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\[ x(i) = \begin{bmatrix} x_1(i) \\ x_2(i) \\ x_3(i) \\ x_4(i) \end{bmatrix} = Q(i) \begin{bmatrix} \bar{x}_{i1} \\ \bar{x}_{i2} \\ \bar{x}_{i3} \\ \bar{x}_{i4} \end{bmatrix}, \quad i \in Z. \tag{2.17} \]

where \( Q(i) \) is defined by (2.13) and the components of the state vector \( \bar{x}(i) \) by (2.6) with \( A_i(i) \) and \( B_i(i) \) defined by (2.14) and (2.16).

3 POSITIVE SYSTEMS

**Definition 3.1.** The descriptor time-varying discrete-time linear system (2.1) is called the (internally) positive if and only if \( x_i \in R^n_+ \) and \( y_i \in R^m_+ \) \( i \in Z \) for any admissible initial conditions \( x_0 \in R^n_+ \) and all inputs \( u_i \in R^n_+ \), \( i \in Z \).

The matrix \( Q(i) \in R^{m \times n}_+ \), \( i \in Z \) is called monomial if in each row and column only one entry is positive and the remaining entries are zero for all \( i \in Z \).

It is well-known (Kaczorek 1998a) that \( Q^{-1}(i) \in R^{n \times m}_+ \), \( i \in Z \) if and only if the matrix is monomial.

It is assumed that for the positive system (2.1) the decomposition (2.3) is positive for the monomial matrix \( Q(i) \). In this case

\[ x_i = Q(i)\bar{x}_i \in R^n_+ \] if and only if \( \bar{x}_i \in R^n_+ \), \( i \in Z. \tag{3.1} \]

It is also well-known that premultiplication of the equation (2.1a) by the matrix \( P(i) \) does not change its solution \( x_i \), \( i \in Z \).

From (2.9) it follows that \( \bar{x}_i \in R^n_+ \), \( i \in Z \) for \( u_i \in R^n_+ \), \( i \in Z \) if and only if

\[ -B_i(i) \in R^{n \times m}_+ \] for \( i \in Z \). \tag{3.2}

In (Kaczorek 2015) has been shown that the time-varying discrete-time system (2.5a) is positive if and only if

\[ A_i(i) \in R^{n \times n}_+, \quad B_i(i) \in R^{m \times n}_+ \], \( i \in Z. \tag{3.3} \]

From (2.1b) and (2.4) we have

\[ y_i = C(i)Q(i)Q^{-1}(i)x_i = C(i)\bar{x}_i, \quad i \in Z. \tag{3.4a} \]

where

\[ C(i) = C(i)Q(i) \tag{3.4b} \]

For monomial matrix \( Q(i) \in R^{m \times n}_+ \) from (3.4) we have \( C(i) \in R^{m \times n}_+ \) if and only if \( C(i) \in R^{m \times n}_+ \), \( i \in Z \) and \( y_i \in R^m_+ \), \( i \in Z \).

Theorem 3.1. The descriptor time-varying discrete-time linear system (2.1) is positive if and only if

1. there exists the decomposition (2.3) for monomial matrix \( Q(i) \in R^{m \times n}_+ \), \( i \in Z \);
2. the conditions (3.2) and (3.3) are satisfied;
3. \( C(i) \in R^{m \times n}_+ \) for \( i \in Z \).

**Example 3.1.** Consider the descriptor time-varying system described by the equation (2.1) with the matrices

\[ E(i) = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2\sin(i) + 4} \\ -\cos(i) - 1 & \frac{1}{\cos(i) + 2} & 0 & -\frac{e^i + 2}{2\sin(i) + 4} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ B(i) = \begin{bmatrix} -\frac{1}{\sin(i) + 2} & 0 \\ e^i + \frac{e^i + 2}{\sin(i) + 2} & -(\cos(i) + 1)(e^i + \sin(i) + 2) \\ 0 & e^i + \sin(i) + 2 \\ 0 & -1 \end{bmatrix} \]

\[ C(i) = \begin{bmatrix} 0 & \frac{1}{\cos(i) + 2} & 0.5 \\ i + 2 & 0 & e^{-i} \\ i + 1 & e^{i + 1} & 0 \end{bmatrix}, \tag{3.5} \]

\[ A(i) = \begin{bmatrix} 0 & 0 & a_{13}(i) & 0 \\ a_{21}(i) & a_{22}(i) & a_{23}(i) & a_{24}(i) \\ a_{31}(i) & 0 & 0 & a_{34}(i) \\ 0 & 0 & 0 & a_{44}(i) \end{bmatrix} \]

where

\[ a_{13}(i) = \frac{1}{(\sin(i) + 2)(e^i + 1)}, \]

\[ a_{21}(i) = -0.3e^i - 0.3\cos(i) - 0.2\sin(i) - 0.3e^i\cos(i) - 0.1, \]

\[ a_{22}(i) = \frac{0.1(i + 1)}{(i + 2)(\cos(i) + 2)}, \]

\[ a_{23}(i) = -\frac{e^i + 2}{(\sin(i) + 2)(e^i + 1)}, \]

\[ a_{24}(i) = \frac{(i + 2)(\cos(i) + 1)(e^i + 1)}{2(i + 1)}, \quad a_{33}(i) = 0.3(e^i + 1), \]

\[ a_{34}(i) = -\frac{(i + 2)(e^i + 1)}{2(i + 1)}, \quad a_{44}(i) = \frac{i + 2}{2(i + 1)}. \]

The condition (2.2) is satisfied since
\[
\det[E(i)A(i)] = e^{-i(3 - 60i + 30i^2 + 3i - 70i^3 + 100i^4 + 200i^5 - 40i^6 + 3)} \quad (3.6)
\]

and
\[
B(i) = \begin{bmatrix} e^{-i} & 0 \\ 0 & 1 + \sin(i) \end{bmatrix} \in R^{2 \times 2}, \quad i \in Z_+
\]

and
\[
C(i) = \begin{bmatrix} 0 & 1 \\ 0 & e^{-i} + 1 \\ i + 1 \\ i + 2 \end{bmatrix} \in R^{4 \times 4} \text{ for } Z_+.
\]

The solution to the equation (2.1) with the matrices \(E(i), A(i), B(i)\) given by (3.5) can be found in a similar way as in Example 2.1.

4 STABILITY OF THE POSITIVE DESCRIPTOR LINEAR SYSTEMS

From (2.1a) and (2.6a) for \(E(i) = I_n, B(i)u_i = 0, \quad i \in Z_+\) it follows that
\[
\hat{x}_{i,i} = \Phi_i(i)\hat{x}_{i,0}, \quad i \in Z_+
\]

where
\[
\Phi_i(i) = \Phi_i(i,0) = A_i(i-1)A_i(i-2)...A_i(0) \quad (4.1b)
\]

is the solution of the equation
\[
\bar{x}_{i,j+1} = A_i(i)\bar{x}_{i,j}, \quad i \in Z_+. \quad (4.2)
\]

From (4.1b) we have
\[
\Phi_i(i+1) = A_i(i)\Phi_i(i), \quad i \in Z_+. \quad (4.3)
\]

**Definition 4.1.** The positive system (4.2) is called asymptotically stable if the norm \(\|\bar{x}_{i,j}^*\|\) of the state vector \(\bar{x}_{i,j}^* \in R^{n_+}_i\) for \(i \in Z_+\) satisfies the condition
\[
\lim_{i \to \infty} \|\bar{x}_{i,0}^*\| = 0 \quad \text{for any finite} \quad \bar{x}_{i,0}^* \in R^{n_+}_i. \quad (4.4)
\]

**Theorem 4.1.** The positive system (4.2) is asymptotically stable if the norm \(\|A_i(i)\|\) of the matrix \(A_i(i)\) for 
\[
\|A_i(i)\| < 1 \quad \text{for} \quad i \in Z_+. \quad (4.5a)
\]

where
\[
\|A_i(i)\| \geq \max_{0 \leq i \leq n_+} \|A_i(i)\| \quad \text{for} \quad i \in Z_+. \quad (4.5b)
\]

Proof is given in (Kaczorek 2015).

**Theorem 4.2.** The positive system (4.2) is asymptotically stable if its system matrix \(A_i(i) = [a_{jk}^{(i)}(i)] \in R^{n_+ \times n_+}\) satisfies the condition
\[
\max_{0 \leq j \leq n} \sum_{k=1}^{n_j} a_{jk}^{(i)}(i) < 1 \text{ for } i \in Z_+ \quad (4.6a)
\]

or

\[
\max_{0 \leq j \leq n} \sum_{k=1}^{n_j} a_{jk}^{(i)}(i) < 1 \text{ for } i \in Z_+ \quad (4.6b)
\]

Proof is given in (Kaczorek 2015).

**Theorem 4.3.** The positive system (4.2) is asymptotically stable if its system matrix

\[
A(i) = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
a_{0}^{(i)}(i) & a_{1}^{(i)}(i) & a_{2}^{(i)}(i) & \ldots & a_{n-1}^{(i)}(i)
\end{bmatrix} \in \mathbb{R}^{n \times n} \quad (4.7)
\]

satisfies the condition

\[
\sum_{k=0}^{n-1} a_{k}^{(i)}(i) < 1 \text{ for } i \in Z_+. \quad (4.8)
\]

Proof is given in (Kaczorek 2015).

Consider the positive descriptor system described by (2.1a) for \( B(i)u_j = 0 \), \( i \in Z_+ \)

\[
E(i)x_{i+1} = A(i)x_i. \quad (4.9)
\]

If the assumption (2.2) is satisfied then the characteristic polynomial of the system (4.9) and of the system

\[
\begin{bmatrix}
I_n & 0 \\
0 & N
\end{bmatrix} x_{i+1} = \begin{bmatrix} A(i) & 0 \\
0 & I_{n_2} \end{bmatrix} x_i \quad (4.10)
\]

are related by

\[
p(z,i) = \det[E(i)z - A(i)] = \det[P^{-1}(i)\begin{bmatrix} I_n & 0 \\
0 & N \end{bmatrix} z - \begin{bmatrix} A(i) & 0 \\
0 & I_{n_2} \end{bmatrix} Q^{-1}(i)] = k(i)\bar{p}(z,i) \quad (4.11a)
\]

where

\[
\bar{p}(z,i) = \det[I_n z - A(i)], \quad k(i) = \det[P^{-1}(i)Q^{-1}(i)] \quad (4.11b)
\]

From (4.11) we have the following lemma.

**Lemma 4.1.** The positive descriptor time-varying discrete-time linear system (4.9) is asymptotically stable if and only if the positive time-varying linear system

\[
\overline{x}_{i+1} = A(i)\overline{x}_i \quad (4.12)
\]

is asymptotically stable.

From Theorem 4.1 and Lemma 4.1 we have the following theorem.

**Theorem 4.4.** The positive descriptor time-varying discrete-time linear system (4.9) is asymptotically stable if the condition

\[
\|A(i)\| = \max_{0 \leq j \leq n} \|A(i)\| < 1 \quad (4.13)
\]

is satisfied.

Similarly, from Theorem 4.2, 4.3 and Lemma 4.1 we have the following theorems.

**Theorem 4.5.** The positive descriptor time-varying discrete-time linear system (4.9) is asymptotically stable if the matrix \( A(i) = [a_{jk}^{(i)}(i)] \in \mathbb{R}^{n \times n}, \ i \in Z_+ \) satisfies the condition

\[
\max_{0 \leq j \leq n} \sum_{k=1}^{n} a_{jk}^{(i)}(i) < 1 \text{ for } i \in Z_+. \quad (4.14a)
\]

or

\[
\max_{0 \leq j \leq n} \sum_{k=1}^{n} a_{jk}^{(i)}(i) < 1 \text{ for } i \in Z_+. \quad (4.14b)
\]

**Theorem 4.6.** The positive descriptor time-varying discrete-time linear system (4.9) is asymptotically stable if the matrix \( A_i(i) \) has the canonical Frobenius form

\[
A(i) = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
a_{0}^{(i)}(i) & a_{1}^{(i)}(i) & a_{2}^{(i)}(i) & \ldots & a_{n-1}^{(i)}(i)
\end{bmatrix} \in \mathbb{R}^{n \times n} \quad (4.15)
\]

and it satisfies the condition

\[
\sum_{k=0}^{n-1} a_{k}^{(i)}(i) < 1 \text{ for } i \in Z_+. \quad (4.16)
\]

**Example 4.1.** (continuation of Example 3.1). By Theorem 4.4 the positive descriptor time-varying discrete-time linear system (2.1) with the matrix \( A(i) \) given by (3.5) is asymptotically stable since
\[ \|A\| = \max_{t \in \mathbb{R}_+} |A(t)| \]

\[ = \max_{t \in \mathbb{R}_+} \left\{ \frac{i+1}{i+2} + 0.2[1 - \sin(i)], \ 0.3(1 - e^{-i}) \right\} < 1 \quad (4.17) \]

for all \( i \in \mathbb{Z}_+ \).

5 CONCLUDING REMARKS

The positivity and asymptotic stability of the descriptor time-varying discrete-time linear systems with regular pencils have been addressed. The Weierstrass-Kronecker theorem on the decomposition of the regular pencils has been extended to the descriptor time-varying discrete-time linear systems. Solutions to the decomposed systems have been derived (Theorem 2.1). Necessary and sufficient conditions for the positivity of the systems have been established (Theorem 3.1). Using the norms of the vectors and matrices sufficient conditions for asymptotic stability of the positive systems have been derived (Theorems 4.1 – 4.6). The effectiveness of the test are demonstrated on examples. The proposed method can be applied in analysis of marine navigation and safety of sea transportation problems. The considerations can be extended to the fractional descriptor time-varying discrete-time linear systems.

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