TRANSNAV International Journal on Marine Navigation and Safety of Sea Transportation
Volume 5 Number 4 December 2011

A Novel Approach to Loxodrome (Rhumb Line), Orthodrome (Great Circle) and Geodesic Line in ECDIS and Navigation in General

A. Weintrit & P. Kopacz
Gdynia Maritime University, Gdynia, Poland

ABSTRACT: We survey last reports and research results in the field of navigational calculations' methods applied in marine navigation that deserve to be collected together. Some of these results have often been rediscovered as lemmas to other results. We present our approach to the subject and place special emphasis on the geometrical base from a general point of view. The geometry of approximated structures implies the calculus essentially, in particular the mathematical formulae in the algorithms applied in the navigational electronic devices and systems. The question we ask affects the range and point in applying the loxodrome (rhumb line) in case the ECDIS equipped with the great circle (great ellipse) approximation algorithms of given accuracy replaces the traditional nautical charts based on Mercator projection. We also cover the subject on approximating models for navigational purposes. Moreover, the navigation based on geodesic lines and connected software of the ship's devices (electronic chart, positioning and steering systems) gives a strong argument to research and use geodesic-based methods for calculations instead of the loxodromic trajectories in general.

1 INTRODUCTION

A common problem is finding the shortest route across the Earth surface between two positions. Such trajectory is always a part of a geodesic (great circle, great ellipse) on the modelling globe surface. The geodesic is used by ship navigators attempting to minimize distances and the radio operators with directional antennae used to look for a bearing yielding the strongest signal. For many purposes, it is entirely adequate to model the Earth as a sphere. Actually, it is more nearly an oblate ellipsoid of revolution. The earth's flattening is quite small, about 1 part in 300, and navigation errors induced by assuming the Earth is spherical do not exceed this, and so for many purposes a spherical approximation may be entirely adequate. On a sphere, the commonly used coordinates are latitude and longitude, likewise on a spheroid, however on a spheroid one has to be more careful about what exactly one means by latitude [Williams, 1996]. The spherical model is often used in cartographic projections creating the frame of the presented chart. The trajectory of the geodesic lines and the loxodrome looks different depending on the method of the projections given by the strict formulae. Thus, many map projections are invaluable in specialized applications.

The only conformal cylindrical projection, Mercator's device was a boon to navigators from the 16th-century until the present, despite suffering from extreme distortion near the poles. We recall it has a remarkable property: any straight line between two points is a loxodrome or line of constant course on the sphere. The Mercator loxodrome bears the same angle from all meridians. Briefly, if one draws a straight line connecting a journey's starting and ending points on a Mercator map, that line's slope yields the journey direction, and keeping a constant bearing is enough to get to one's destination.

A Mercator projection is not the only one used by navigators, as the loxodrome does not usually coincides with the geodesic. This projection was possibly first used by Etzlaub (ca. 1511). However, it was for sure only widely known after Mercator's atlas of 1569. Mercator probably defined the graticule by geometric construction. E. Wright formally presented equations in 1599. Wright's work influenced, among other persons, Dutch astronomer and mathematician Willebrord Snellius, who introduced the

word "loxodrome"; Adriaan Metius, the geometer and astronomer from Holland; and the English mathematician Richard Norwood, who calculated the length of a degree on a great circle of the Earth using a method proposed by Wright.

More commonly applied to large-scale maps, the transverse aspect preserves every property of Mercator's projection, but since meridians are not straight lines, it is better suited for topography than navigation. Equatorial, transverse and oblique maps offer the same distortion pattern. The transverse aspect with equations for the spherical case was presented by Lambert in his seminal paper (1772). The ellipsoidal case was developed, among others, by Carl Gauss (ca. 1822) and Louis Krüger (ca. 1912). It is frequently called the Gauss conformal or Gauss-Krüger projection.

The vessel or aircraft can reach its destination following the fixed bearing along the whole trip disregarding some obvious factors like for instance weather, fuel range, geographical obstructions. However, that easy route would not be the most economical choice in terms of distance. The two paths almost coincide only in brief routes. Although the rhumb line is much shorter on the Mercator map, an azimuthal equidistant map tells a different story, even though the geodesic does not map to a straight line since it does not intercept the projection centre. Since there is a trade-off: following the geodesic would imply constant changes of direction (those are changes from the current compass bearing and are only apparent: on the sphere, the trajectory is as straight as it can be). Following the rhumb line would waste time and fuel. So a navigator could follow a hybrid procedure [Snyder, 1987]:

- trace the geodesic on an azimuthal equidistant or gnomonic map,
- break the geodesic in segments,
- plot each segment onto a Mercator map,
- use a protractor and read the bearings for each segment,
- navigate each segment separately following its corresponding constant bearing.

2 GEODESIC APPROACH

For curved or more complicated modelling surfaces the metric can be used to compute the distance between two points by integration. The distance generally means the shortest distance between two points. Roughly speaking, the distance between two points is the length of the path connecting them. Most often the research and calculus in navigational literature are considered on the spherical or spheroidal models of Earth because of practical reasons. The flow of geodesics on the ellipsoid of revolution (spheroid) differs from the geodesics on the sphere. There are known different geodesics on the same surface with the same metric considered. However geodesic refers to the metric what is usually not taken into consideration in the navigational lectures. And there are different flows of geodesics on the same surface when different metrics are applied. That means we can obtain geometrically different results in navigational aspect if we change the researched modelling object with its geometrical and physical features (Kopacz, 2006).

Let us focus on two essential notions creating the base for the various fields of the mathematical research: the metric and topology. A metric space is a set with a global distance function (the metric) that, for every two points in, gives the distance between them as a nonnegative real number.

Definition 1. A function $g: X \times X \rightarrow [0, \infty)$ is called *a metric* (or distance) in X if

(1) $g(x, y) = 0$ iff $x = y$ (positivity);

(2) $g(x, y) = g(y, x)$ for every $x, y \in X$ (symmetry);

(3) $g(x, y) \leq g(x, z) + g(z, y)$ for every $x, y, z \in X$ (triangle inequality).

Metric as a nonnegative function describes the "distance" between neighbouring points for a given set. When viewed as a tensor, the metric is called a metric tensor. We can define a metric in each non-empty set ($X \neq \emptyset$). The notion of metric has been introduced by M. Frechet in 1906. Formally the pair (X, g) where g is a metric in a set X is called a *metric space*. Fig. 1 points out the essential role played by the metric in geodesic approach to the subject.

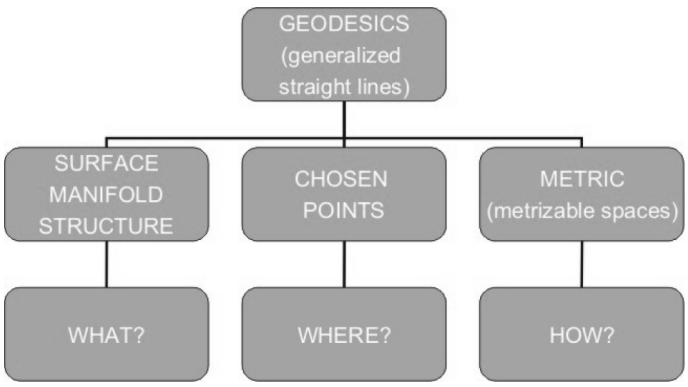


Figure 1. Geometrical basis in geodesic analysis [Kopacz, 2006]

Making one step further we can generalize the metric space to the topological space.

Definition 2. Let $X \neq \emptyset$ be a set and $P(X)$ the power set of X, i.e. $P(X) = \{U : U \subset X\}$. Let $\Omega \subset P(X)$ be a collection of its subsets such that:

(1) $(\forall \iota \in \mathrm{I} \quad U_\iota \in \Omega) \Rightarrow \bigcup_{\iota \in \mathrm{I}} U_\iota \in \Omega$ (the union of a collection of sets, which are elements of Ω, belongs to Ω);

(2) $U, V \in \Omega \Rightarrow U \cap V \in \Omega$ (the intersection of a finite collection of sets, which are elements of Ω, belongs to Ω);

(3) $\varnothing$, $X \in \Omega$, (the empty set $\varnothing$ and the whole set X belong to Ω).

Then

- Ω is called a topological structure or just a topology in X;
- the pair (X, Ω) is called a topological space;
- an element of X is called a point of this topological space;
- an element of Ω is called an open set of the topological space (X, Ω).

The conditions in the definition presented above are called the *axioms of topological structure.* A topology, that is a metric topology, means that one can define a suitable metric that induces it. Additionally we assume here that although the metric exists, it may be unknown. In a metric space (X, g) the family of sets Ω

$$\Omega = \{U \subset X : \underset{x \in U}{\forall} \; \underset{\varepsilon > 0}{\exists} \; B(x, \varepsilon) \subset U\}$$

satisfies the above mentioned axioms of topology. That means (X, g) is a topological space and thus, each metric space is a topological space. There are sufficient criteria on the topology that assure the existence of such a metric even if this is not explicitly given. An example of an existence theorem of this kind is due to Urysohn who proved that a regular T_1-space whose topology has a countable basis is metrizable [Kelley, 1955]. Conversely, a metrizable space is always T_1 and regular but the condition on the basis has to be weakened since in general, it is only true that the topology has a basis which is formed by countably many locally finite families of open sets. Special metrizability criteria are known for Hausdorff spaces (T_2-spaces). A compact Hausdorff space is metrizable if and only if the set of all elements is a zero set [Willard, 1970]. The continuous image of a compact metric space in a Hausdorff space is metrizable. This implies in particular that a distance can be defined on every path in T_2-space.

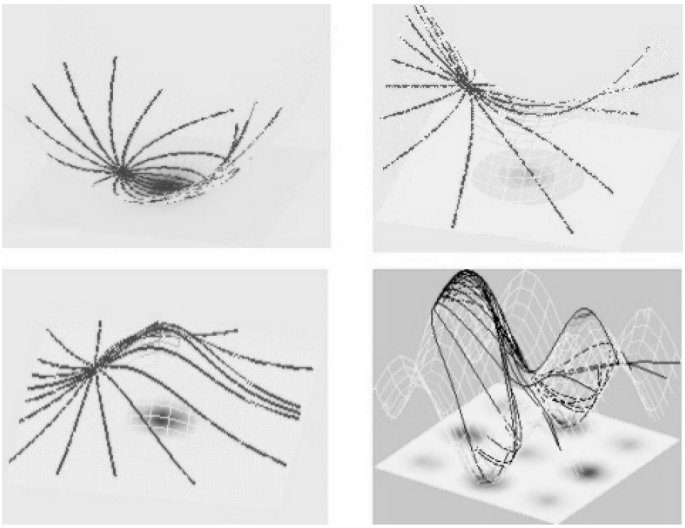


Figure 2. Flows of geodesics (distance functions) on locally modelling surfaces of differing curvatures

The mathematical formulae used in approximation of the navigational calculations are being studied and are based on spherical (spheroidal) model. However if we consider different shape of the surface the formulae change considerably. The examples of the flows of geodesics on locally modelling surfaces of differing curvature are presented graphically in Fig. 2. Let us imagine that the vessels do not sail on spheroidal earth but locally torus - shaped planet. In this case the flow of geodesics and mentioned rhumb line or used charts are based on other mathematical expressions due to different geometrical object considered. The torus is topologically more simple than the sphere, yet geometrically it is a very complicated manifold indeed.

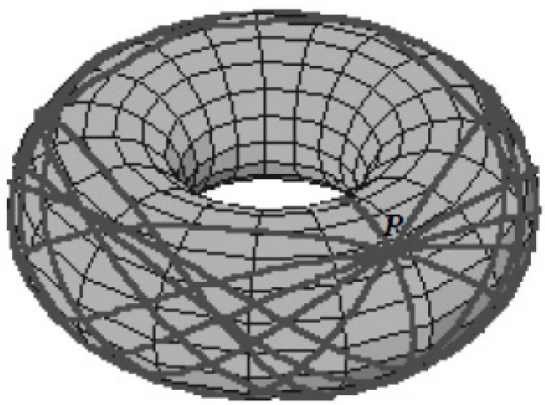


Figure 3. The geodesics on a torus $T^2 = S^1 \times S^1$

The round torus metric is most easily constructed via its embedding in a Euclidean space of one higher dimension.

Taking into consideration the main theoretical aspects of the subject above mentioned as well as the practical ones influencing the base and components of the navigational algorithm to be applied we col-

lect all of them together what has been shown in Figure 4.

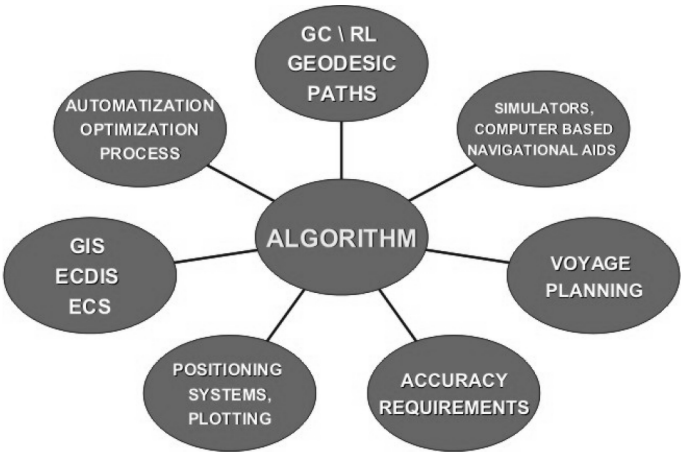


Figure 4. Navigational calculations' algorithm guidelines

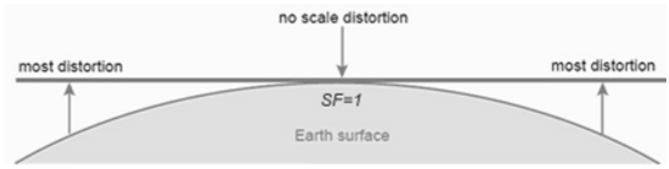
The notion of geodesics makes sense not only for surfaces in R^3 but also for abstract surfaces and more generally (Riemannian) manifolds. We also refer to [Funar, Gadgil, 2001] where the notion of a topological geodesic in a *3*-manifold have been introduced. Geodesics in Riemannian manifolds with metrics of negative sectional curvature play an essential role in geometry. It is shown there that, in the case of *3*-dimensional manifolds, many crucial properties of geodesics follow from a purely topological characterization in terms of *knotting* as well as proved basic existence and uniqueness results for topological geodesics under suitable hypotheses on the fundamental group. For further reading we send the reader to the wide literature on Riemannian and Finsler geometry and topology, in particular the geodesic research.

3 PLANE MODEL

The surface of revolution as the Earth's model - sphere S^2 or the spheroid is locally approximated by the Euclidean plane tangent in a given position. Generally, we approximate locally the curved surface by the Euclidean plane. For some applications such approximation is allowed and sufficient for practical need of research. That is satisfactory if we do not exceed the required accuracy of provided calculations. Hence the boundary conditions of applying the Euclidean plane or spherical geometry ought to be strictly defined. The mathematical components of the plane Euclidean geometry applied in navigational device are widely known and there is a common Euclidean metric used in the calculus as the distance function. We emphasize that the geodesics may look different even on the plane if different metrics are considered. For the practical reasons and the ease of use there is Euclidean plane tangent to the modelled surface used in many applications, for instance in dynamic positioning (DP) software. The plane model enables the satisfactory accuracy in a local approximation. In the local terrain geodesic research the area can be considered flat if it is inside the circle of a radius of ca. 15.5 km. This corresponds to the area of spherical circle which diameter equals ca. 17' of the great circle [Kopacz, 2010]. Practically such an approximation allows the direct geodesic measurements without considering the curvature of the modelled Earth surface and presenting the results on the plane in the appropriate scale. In the global modelling of the Earth's surface (geodesy, cartography, navigation, astronomy) the Euclidean geometry becomes not sufficient for the geometric description and the calculus coming from it. Thus, the limits of application of the approximation methods based on the flat Euclidean geometry must be clearly determined [Kopacz, 2010].

In a field of flat chart projections scale distortions on a chart can be shown by means of ratio of the scale at a given point to the true scale (*a scale factor - SF*). Scale distortions exist at locations where the scale factor differs from 1. For instance, a scale factor at a given point on the map is equal to 0.99960 signifies that 1000 m on the reference surface of the Earth will actually measure 999.6 m on the chart. This is a contraction of 40 cm per 1 km.

a)



b)

Figure 5. Scale distortions on a tangent (a) and a secant (b) map surface [Knippers, 2009]

Distortions increase as the distance from the central point (tangent plane) or closed line(s) of intersection increases. Scale distortions for tangent and secant map surfaces are illustrated in the Fig. 5. On a secant map projection - the application of a scale factor of less than 1.0000 to the central point or the central meridian has the effect of making the projection secant - the overall distortions are less than on one that uses a tangent map surface. Most countries have derived there map coordinate system from a projection with a secant map surface for this reason [Knippers, 2009].

The curved Earth is navigated using flat maps or charts, collected in an atlas. Similarly, in a calculus on manifolds a differentiable manifold can be described using mathematical maps, called coordinate charts, collected in a mathematical atlas. It is not generally possible to describe a manifold with just one chart, because the global structure of the manifold is different from the simple structure of the charts. For example, no single flat map can properly represent the entire Earth. When a manifold is constructed from multiple overlapping charts, the regions where they overlap carry information essential to understanding the global structure. In the case of a differentiable manifold, an atlas allows to do calculus on manifolds. The atlas containing all possible charts consistent with a given atlas is called the maximal atlas. Unlike an ordinary atlas, the maximal atlas of a given atlas is unique. Though it is useful for definitions, it is a very abstract object and not used directly for example in calculations. Charts in an atlas may overlap and a single point of a manifold may be represented in several charts. If two charts overlap, parts of them represent the same region of the manifold. Given two overlapping charts, a transition function can be defined which goes from an open ball in R^n to the manifold and then back to another (or perhaps the same) open ball in R^n. The resultant map is called a change of coordinates, a coordinate transformation, a transition function or a transition map.

4 SPHERICAL AND SPHEROIDAL MODEL

As the Earth's global model an oblate spheroid is used providing the navigational calculations i.e. distances and angles or the sphere for the ease of use. A sphere, spheroid or a torus surface are examples of *2*-dimensional manifolds. Manifolds are important objects in mathematics and physics because they allow more complicated structures to be expressed and understood in terms of the relatively well understood properties of simpler spaces. The study of manifolds combines many important areas of mathematics: it generalizes concepts such as curves and surfaces as well as ideas from linear algebra and topology. Certain special classes of manifolds also have additional algebraic structure. They may behave like groups, for instance. To measure distances and angles on manifolds, the manifold must be Riemannian. We recall that a Riemannian manifold is an analytic manifold in which each tangent space is equipped with *an inner product* in a manner which varies smoothly from point to point. This allows one to define various notions such as length, angles, areas (or volumes), curvature, gradients of functions and divergence of vector fields. More general geometric structure a Finsler manifold allows the definition of distance, but not of angle. It is an analytic manifold in which each tangent space is equipped with *a norm*, in a manner which varies smoothly from point to point. This norm can be extended to a metric, defining the length of a curve; but it cannot in general be used to define an inner product. Any Riemannian manifold is a Finsler manifold. Manifold theory has come to focus exclusively on these intrinsic properties (or invariants), while largely ignoring the extrinsic properties of the ambient space.

Triaxial ellipsoid as the *2*-dimensional submanifold M in R^3 is defined by the equation $\Phi = 0$ where

$$\Phi(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 .$$

Let N be the non-vanishing normal vector field on M. Then

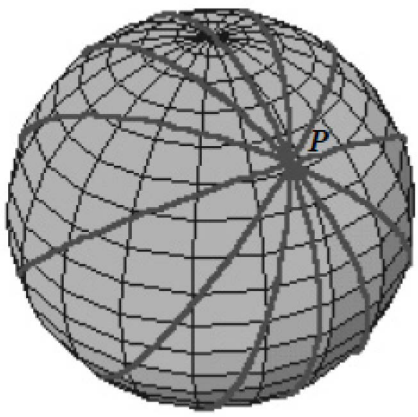
$$N(x,y,z) = 0,5 grad\Phi = \frac{x}{a^2}e_1 + \frac{y}{b^2}e_2 + \frac{z}{c^2}e_3$$

where the $\{e_1, e_2, e_3\}$ is the canonical basis of the vector space R^3. The Gaussian curvature K of the modelling triaxial ellipsoid equals $K = \frac{1}{a^2b^2c^2\|N\|^4}$.

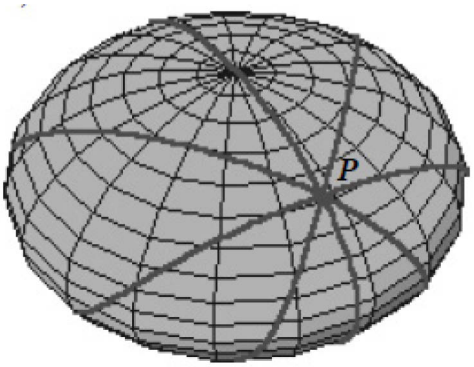
The Gaussian curvature is the determinant of the shape operator. For the sphere $a=b=c=r$ and then $\|N\| = \frac{1}{r}, K = \frac{1}{r^2}$ where r denotes a radius of the modelling sphere. Thus, we conclude here that the curvature affects the geometry of the locally approximating surfaces essentially and in particular their geodesic trajectories.

2-dimensional sphere S^2 is widely considered to model globally the surface of the Earth. As a calculating tool the spherical trigonometry is used which states the base for the comparison analysis and algorithms implemented in the software of navigational aids e.g. receivers of the positioning systems, ECDIS. The surface of the Earth may be taken mathematically as a sphere instead of ellipsoid for maps at smaller scales. In practice, maps at scale 1:5 000 000 or smaller can use the mathematically simpler sphere without the risk of large distortions. At larger scales, the more complicated mathematics of ellipsoids are needed to prevent these distortions in the map. A sphere can be derived from the certain ellipsoid corresponding either to the semi-major or semi-minor axis, or average of both axes or can have equal volume or equal surface than the ellipsoid [Knippers, 2009].

a)



b)



c)

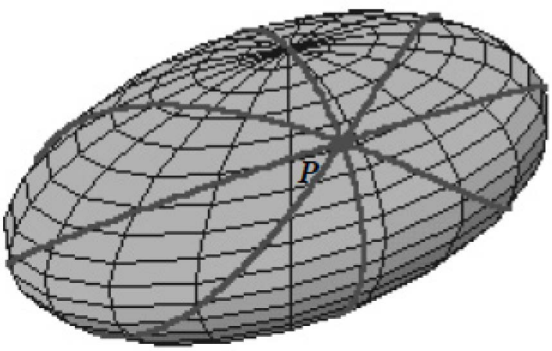


Figure 6. Geodesics on *2*-dimensional modelling manifolds of positive curvature a) sphere, b) spheroid (ellipsoid of revolution), c) triaxial ellipsoid.

We recall the great circle is the equivalent of the Euclidean straight line, it has the finite distance and it is closed. The geodesics starting from a given position on three main modelling surfaces (*2*-dimensional modelling manifolds of positive curvature), i.e. sphere, spheroid and triaxial ellipsoid are presented in Fig. 6. The disadvantage of orthodromic sailing is bound with continuous course alteration. That is why the loxodromic line is mainly sailed only or mainly used in the approximation of the great circle sailing. Thus the combination of these two lines create the base for planned and monitored trajectories while at sea.

The general question we ask affects the range and point of usage of the rhumb line in case the ECDIS systems equipped with the great circle / great ellipse approximation algorithms of given accuracy replaces the traditional paper charts based on Mercator projection. Moreover, the navigation based on geodesic lines and connected software of the ship's device (electronic chart, positioning and steering systems) gives a strong argument to use this method for calculations instead of the loxodromic one in general. Although the basic solutions for navigational purposes have already been known and widely used there are still the new approaches and efforts made to the subject. The examples of the spherical and spheroidal approach have been found recently in the literature reviewed further in the article. The main efforts affect the optimization and approximation methods which potentially may give the practical benefits for the navigators.

As we mentioned above the shortest path between two points on a smooth surface is called a geodesic curve on the surface. On a flat surface the geodesics are the straight lines, on a sphere they are the great circles. Remarkably the path taken by a particle sliding without friction on a surface will always be a geodesic. This is because a defining characteristic of a geodesic is that at each point on its path, the local centre of curvature always lies in the direction of the surface normal, i.e. in the direction of any constrained force required to keep the particle on the surface. There are thus no forces in the local tangent plane of the surface to deflect the particle from its geodesic path. There is a general procedure, using the calculus of variations, to find the equation for geodesics given the metric of the surface [Williams, 1996]. Obviously the Earth is not an exact ellipsoid and deviations from this shape are continually evaluated. The *geoid* is the name given to the shape that the Earth would assume if it were all measured at mean sea level. This is an undulating surface that varies not more than about a hundred meters above or below a well-fitting ellipsoid, a variation far less than the ellipsoid varies from the sphere. The choice of the reference ellipsoid used for various regions of the Earth has been influenced by the local geoid, but large-scale map projections are designed to fit the reference ellipsoid, not the geoid. The selection of constants defining the shape of the reference ellipsoid has been a major concern of geodesists since the early 18th century. Two geometric constants are sufficient to define the ellipsoid itself e.g. the semimajor axis and the eccentricity. In addition, recent satellite-measured reference ellipsoids are defined by the semimajor axis, geocentric gravitational constant and dynamical form factor which may be converted to flattening with formulas from physics.

Between 1799 and 1951 there were 26 determinations of dimensions of the Earth. There are over a

dozen other principal ellipsoids, however, which are still used by one or more countries. The different dimensions do not only result from varying accuracy in the geodetic measurements (the measurements of locations on the Earth), but the curvature of the Earth's surface (geoid) is not uniform due to irregularities in the gravity field. Until recently, ellipsoids were only fitted to the Earth's shape over a particular country or continent. The polar axis of the reference ellipsoid for such a region, therefore, normally does not coincide with the axis of the actual Earth, although it is assumed to be parallel. The same applies to the two equatorial planes. The discrepancy between centres is usually a few hundred meters at most. Satellite-determined coordinate systems are considered geocentric. Ellipsoids for the latter systems represent the entire Earth more accurately than ellipsoids determined from ground measurements, but they do not generally give the best fit for a particular region. The reference ellipsoids used prior to those determined by satellite are related to an initial point of reference on the surface to produce a datum, the name given to a smooth mathematical surface that closely fits the mean sea-level surface throughout the area of interest. The initial point is assigned a latitude, longitude, elevation above the ellipsoid, and azimuth to some point. Satellite data have provided geodesists with new measurements to define the best Earth-fitting ellipsoid and for relating existing coordinate systems to the Earth's centre of mass. For the mapping of other planets and natural satellites, Mars is treated as an ellipsoid. Other bodies are taken as spheres, although some irregular satellites have been treated as triaxial ellipsoids and are mapped orthographically [Snyder, 1987].

5 ECDIS APPROACH

In the course of navigation programmes for ECDIS purposes it became apparent that the standard text books of navigation were perpetuating a flawed method of calculating rhumb lines on the Earth considered as an oblate spheroid. On further investigation it became apparent that these incorrect methods were being used in programming a number of calculator/computers and satellite navigation receivers. Although the discrepancies were not large, it was disquieting to compare the results of the same rhumb line calculations from a number of such devices and find variations of a few miles when the output was given, and therefore purported to be accurate, to a tenth of a mile in distance and/or a tenth of a minute of arc in position. The problem was highlighted in the past and the references at the end of this paper show that a number of methods have been proposed for the amelioration of this problem.

This paper presents and recommends the guidelines that should be used for the accurate solutions. Most of these may be found in standard geodetic text books, such as, but also provided are new formulae and schemes of solution which are suitable for use with computers or tables. The paper also takes into account situations when a near-indeterminate solution may arise. Some examples are provided which demonstrate the methods. The data for these problems do not refer to actual terrestrial situations but have been selected for illustrative purposes. Practising ships' navigators will find the methods described in detail in this paper to be directly applicable to their work and they also should find ready acceptance because they are similar to current practice. In almost none of the references cited at the end of this paper has been addressed the practical task of calculating, using either a computer or tabular techniques.

The paper presents the review of different approaches to contact formulae for the computation of the position, the distance, and the azimuth along a great ellipse. The proposed alternative formulae are to be primarily used for accurate sailing calculations on the ellipsoid in a GIS environment as in ECDIS and other ECS. Among the ECDIS requirements is the need for a continuous system with a level of accuracy consistent with the requirements of safe navigation. At present, this requirement is best fulfilled by the Global Positioning System (GPS). The GPS system is referenced to World Geodetic System 1984 Datum (WGS 84). Using the ellipsoid model instead of the spherical model attains more accurate calculation of sailing on the Earth. Therefore, we aim to construct a computational procedure for solving the length of the arc of a great ellipse, the waypoints and azimuths along a great ellipse. We announce our aspiration to provide the straightforward formulae involving the great elliptic sailing based on two scenarios. The first is that the departure point and the destination point are known. The second is that the departure point and the initial azimuth are given (direct and inverse geodetic problems on reference ellipsoids).

5.1 *ECDIS Calculations*

As a minimum, an ECDIS system must be able to perform the following calculations and conversions [Weintrit, 2009]:

- geographical coordinates to display coordinates, and display coordinates to geographical coordinates;
- transformation from local datum to WGS-84;
- true distance and azimuth between two geographical positions;
- geographic position from a known position given distance and azimuth (course);

- projection calculations such as great circle and rhumb line courses and distances;
- "RL-GC" difference between the rhumb line and great circle in sailing along the great circle (or great ellipse?).

5.2 *Route planning calculations*

The ECDIS allows the navigator to create waypoints and routes including setting limits of approach and other cautionary limits. Both rhumb line and great circle routes can be defined. Routes can be freely exchanged between the ECDIS and GPS or ARPA. Route checking facility allows the intended route to be automatically checked for safety against limits of depth and distance as defined by the navigator.

The mariner can calculate and display both a rhumb line and a great circle line and verify that no visible distortion exists between these lines and the chart data.

Authors predict the early end of the era of the rhumb line. This line in the natural way will go out of use. Nobody after all will be putting the navigational triangle to the screen of the ECDIS. Our planned route is not having to be a straight line on the screen. So, why hold this line still in the use? Each ship's position plotted on the chart can be the starting point of new updated great circle GC, or saying more closely, great ellipse GE.

5.3 *Most important questions*

It is an important question whether in the ECDIS time still Mercator projection is essential for marine navigation. We really need it? And what about loxodrome? Also not? So, let start navigation based on geodesics. It is high time to forget the rhumb line navigation and great circle navigation, too. But the first we need clear established methods, algorithms and formulas for sailing calculations. But it is already indicating the real revolution in navigation - total revolution. We will be forced to make the revision of such fundamental notions as the course, the heading and the bearing.

And another very important question: do you really know what kind of algorithms and formulae are used in your GPS receiver and your ECS/ECDIS systems for calculations mentioned in chapter 5.1? We are sure, your answer is negative. So, we have a problem – a serious problem.

6 REVIEW OF RECENTLY PUBLISHED PAPERS

From 1950 till 2010 the following professional magazines and journals published some papers about navigation on the great ellipse and on the spheoridical Earth: *The Journal of Navigation* [Bennett, 1996; Bourbon, 1990; Carlton-Wippern, 1992; Chen, Hsu, & Chang, 2004; Earle, 2000, 2005, 2006, 2008; Hickley, 1987; Hiraiwa, 1987; Nastro & Tancredi, 2010; Pallikaris & Latsas, 2009; Prince & Williams, 1995; Sadler, 1956; Tseng & Lee, 2007; Tyrrell, 1955; Walwyn, 1999; Williams, 1950; Williams, 1996; Williams & Phythian, 1989, 1992; Zukas, 1994], *International Hydrographic Review* [Pallikaris, Tsoulos & Paradissis, 2009a], *Coordinates* [Pallikaris, Tsoulos & Paradissis, 2010], *Navigation - The Journal of The Institute of Navigation* [Kaplan, 1995; Miller, Moskowitz & Simmen, 1991], *Bulletin Geodesique* [Bowring, 1983, 1984; Rainsford, 1953, 1955; Sodano, 1965], *Journal of Marine Science and Technology* [Tseng & Lee, 2010], *The Journal of the Washingtin Academy of Sciences* [Lambert, 1942], *The Canadian Surveyor* [Bowring, 1985], *Survey Review* [Vincenty, 1975, 1976], *Surveying and Mapping* [Meade, 1981], *The Professional Geographer* [Tobler, 1964], *College Mathematics Journal* [Nord, Muller, 1996; Schechter, 2007].

The following particular problems were discussed among the others:

- practical rhumb line calculations on the spheroid [Bennet, 1996],
- geodesic inverse problem [Bowring, 1983],
- direct and inverse solutions for the great elliptic and line on the reference ellipsoid [Bowring, 1984],
- loxodromic navigation [Carlton-Wippern, 1992],
- formulas for the solution of direct and inverse problems on reference ellipsoids using pocket calculators [Meade, 1981],
- geometry of loxodrome on the ellipsoid [Bowring, 1985],
- geometry of geodesics [Busemann, 1955],
- geodesic line on the surface of a spheroid [Bourbon, 1990],
- great circle equation [Chen, Hsu & Chang, 2004],
- novel approach to great circle sailing [Chen, Hsu & Chang, 2004],
- vector function of traveling distance for great circle navigation [Tseng & Lee, 2007],
- great circle navigation with vectorial methods [Nastro & Tancredi, 2010],
- vector solution for great circle navigation [Earle, 2005],
- vector solution for navigation on a great ellipse [Earle, 2000],
- navigation on a great ellipse [Tseng & Lee, 2010],

- great ellipse solution for distances and headings to steer between waypoints [Walwyn, 1999],
- great ellipse on the surface of the spheroid [Williams, 1996],
- vector solutions for azimuth [Earle, 2008],
- sphere to spheroid comparisons [Earle, 2006],
- great circle versus rhumb line cross-track distance at mid-longitude [Hickley, 1987],
- modification of sailing calculations [Hiraiwa, 1987],
- practical sailing formulas for rhumb line tracks on an oblate Earth [Kaplan, 1995],
- distance between two widely separated points on the surface of the Earth [Lambert, 1942],
- traveling on the curve Earth [Miller, Moskowitz & Simmen, 1991],
- new meridian arc formulas for sailing calculations in GIS [Pallikaris, Tsoulos & Paradissis, 2009a],
- new calculations algorithms for GIS navigational systmes and receivers [Pallikaris, Tsoulos & Paradissis, 2009b],
- improved algorithms for sailing calculations [Pallikaris, Tsoulos & Paradissis, 2010],
- new algorithm for great elliptic sailing (GES) [Pallikaris & Latsas, 2009],
- shortest paths [Lyusternik, 1964],
- sailing in ever-decreasing circles [Prince & Williams, 1995],
- long geodesics on the ellipsoid [Rainsford, 1953, 1955],
- spheroidal sailing and the middle latitude [Sadler, 1956],
- general non-iterative solution of the inverse and direct geodetic problems [Sodano, 1965],
- comparison of spherical and ellipsoidal measures [Tobler, 1964],
- navigating on the spheroid [Tyrrell, 1955; Williams, 2002],
- direct and inverse solutions of geodesics on the ellipsoid with application of nested equations [Vincenty, 1975, 1976],
- loxodromic distances on the terrestrial spheroid [Williams, 1950],
- Mercator's rhumb lines: a multivariable application of arc length [Nord, Muller, 1996],
- navigating along geodesic paths on the surface of a spheroid [Williams & Phythian, 1989],
- shortest distance between two nearly antipodean points on the surface of a spheroid [Williams & Phythian, 1992],
- shortest spheroidal distance [Zukas, 1994],
- navigating on a spheroid [Schechter, 2007].

7 CONCLUSIONS

This article is written with a variety of readers in mind, ranging from practising navigators to theoretical analysts. It was also our goal to present a current and uniform approaches to sailing calculations highlighting recent developments. Much insight may be gained by considering the examples that have recently proliferated in the literature reviewed above. We present our approach to the subject and place special emphasis on the geometrical base from a general point of view. Of particular interest are geodesic lines, in particular great ellipse calculations. The geometry of modelling structures implies the calculus essentially, in particular the mathematical formulae in the algorithms applied in the navigational electronic device and systems. Thus, is the spherical or spheroidal model the best fit in the local approximations of the Earth surface? We show that generally in navigation the essential calculating procedure refers to the distance and angle measurement what may be transferred to more general geometrical structures, for instance metric spaces, Riemannian manifolds. The authors point out that the locally modelling structure has a different "shape" and thus the different curvature and the flow of geodesics. That affects the calculus provided on it. The algorithm applied for navigational purposes, in particular ECDIS should inform the user on actually used mathematical model and its limitations. The question we also ask affects the range and point in applying the loxodrome sailing in case the ECDIS equipped with the great circle (great ellipse) approximation algorithms of given accuracy replaces the traditional nautical charts based on Mercator projection. The shortest distance (geodesics) depends on the type of metric we use on the considered surface in general navigation. The geodesics can look different even on the same plane if different metrics are taken into consideration. Let us observe for instance the diameter of the parallel of latitude conical circle does not pass its centre. That differs from both the plane and spherical case. Our intuition insists on the way of thinking to look at the diameter as a part of geodesic of the researched surface crossing the centre of a circle. However the diameter depends on the applied metric, thus the shape of the circles researched in the metric spaces depends on the position of the centre and the radius. It is also important to know how the distance between two points on considered structure is determined, where the centre of the circle is positioned and how the diameter passes. Changing the metric causes the differences in the obtained distances. For example π as a number is constant and has the same value in each geometry it is used in calculations. However π as a ratio of the circumference to its diameter can achieve different values in general, in particular π [Kopacz, 2010]. The navigation based on geodesic lines and connected software of the ship's devices (electronic chart, positioning and steering systems) gives a strong argument to research and use geodesic-based methods for calculations instead of the loxodromic trajectories in gen-

eral. The theory is developing as well what may be found in the books on geometry and topology. This motivates us to discuss the subject and research the components of the algorithm of calculations for navigational purposes.

REFERENCES

Admiralty Manual of Navigation, 1987, Volume 1, General navigation, Coastal Navigation and Pilotage, Ministry of Defence (Navy), London, The Stationery Office.

Bennett, G.G. 1996. *Practical rhumb line calculations on the* spheroid, The Journal of Navigation, Vol. 49, No. 1.

Bowditch, N. 2002. *American Practical Navigator*, Pub. No. 9, The Bicentennial Edition, National Imagery and Mapping Agency.

Bowring, B.R. 1983. *The Geodesic Inverse Problem*, Bulletin Geodesique, Vol. 57, p. 109 (Correction, Vol. 58, p. 543).

Bowring, B.R. 1984. *The direct and inverse solutions for the great elliptic and line on the reference ellipsoid*, Bulletin Geodesique, 58, p. 101-108.

Bowring, B.. 1985. *The geometry of Loxodrome on the Ellipsoid*, The Canadian Surveyor, Vol. 39, No. 3.

Busemann, H. 1955. *The geometry of geodesics*, Elsevier Academic Press, Washington.

Bourbon, R. 1990. *Geodesic line on the surface of a spheroid*, The Journal of Navigation, Vol. 43, No. 1.

Carlton-Wippern, K. 1992. *On Loxodromic Navigation*, The Journal of Navigation, Vol. 45, No. 2., p.292-297.

Chen, C.L., Hsu, T.P., Chang, J.R. 2004. *A Novel Approach to Great Circle sailing: The Great Circle Equation*, The Journal of Navigation, Vol. 57, No. 2, p. 311-325.

Cipra, B. 1993. *What's Happening in the Mathematical Sciences*, Vol. I. RI: American Mathematical Society, Providence.

Earle, M.A. 2000. Vector Solution for Navigation on a Great Ellipse, The Journal of Navigation, Vol. 53, No. 3., p. 473-481.

Earle, M.A. 2005. *Vector Solution for Great Circle Navigation*, The Journal of Navigation, Vol. 58, No. 3, p. 451-457.

Earle, M.A. 2006. *Sphere to spheroid comparisons*, The Journal of Navigation, Vol. 59, No. 3, p. 491-496.

Earle, M.A. 2008. *Vector Solutions for Azimuth*, The Journal of Navigation, Vol. 61, p. 537-545.

Funar, L., Gadgil, S. 2001. *Topological geodesics and virtual rigidity*, Algebraic & Geometric Topology, Vol. 1, pp. 369-380.

Goldberg, D. 1991. *What Every Computer Scientist Should Know About Floating-Point Arithmetic*, ACM Computing Surveys, March.

Gradshteyn, I.S., & Ryzhik, I.M.. 2000. *Tables of Integrals, Series and Products*, 6th ed. CA Academic Press, San Diego.

Hickley, P. 1987. *Great Circle Versus Rhumb Line Cross-track Distance at Mid-Longitude*. The Journal of Navigation, Vol. 57, No. 2, p. 320-325 (Forum), May.

Hiraiwa, T. 1987. *Proposal on the modification of sailing calculations*. The Journal of Navigation, Vol. 40, 138 (Forum).

Hohenkerk, C., & Yallop B.D. 2004. *NavPac and Compact Data 2006 – 2010 Astro-Navigation Methods and Software for the PC*. TSO, London.

Kaplan, G.H. 1995. *Practical Sailing Formulas For Rhumb-Line Tracks on an Oblate Earth,* Navigation, Vol. 42, No. 2, pp. 313-326, Summer 1995;

Kelley, J. L. 1955. *General Topology.* New York: Van Nostrand.

Knippers, R. 2009. *Geometric aspects of mapping*, International Institute for Geo-Information Science and Earth Observation, Enschede, http://kartoweb.itc.nl/geometrics.

Kopacz, P. 2006. On notion of foundations of navigation in maritime education, 7th Annual General Assembly and Conference AGA-7, The International Association of Maritime Universities IAMU, Dalian, China.

Kopacz, P. 2007. On Zermelo navigation in topological structures, 9th International Workshop for Young Mathematicians "Topology", pp. 87–95, Institute of Mathematics, Jagiellonian University, Cracow.

Kopacz, P. 2010. Czy pi jest constans?, „Delta" - Matematyka - Fizyka – Astronomia – Informatyka, No 12/2010, Instytut Matematyki, Uniwersytet Warszawski, Warszawa (in Polish).

Lambert, W.D. 1942. *The Distance Between Two Widely Separated Points on the Surface of the Earth*, The Journal of the Washingtin Academy of Sciences, Vol. 32, No. 5, p. 125, May.

Lyusternik, L.A. 1964. *Shortest Paths*, Pergamon Press.

Meade, B.K. 1981. *Comments on Formulas for the Solution of Direct and Inverse Problems on Reference Ellipsoids using Pocket Calculators*, Surveying and Mapping, Vol. 41, March.

Miller, A.R., Moskowitz, I.S., Simmen J. 1991. *Traveling on the Curve Earth*, The Journal of The Institute of Navigation, Vol. 38.

Nastro, V., Tancredi U. 2010. *Great Circle Navigation with Vectorial Methods*, Journal of Navigation, Vol.63, Issue 3.

Nord, J., Muller, E. 1996. *Mercator's rhumb lines: A multivariable application of arc length*, College Mathematics Journal, No. 27, p. 384–387.

Pallikaris, A., Tsoulos, L., Paradissis, D. 2009a. *New meridian arc formulas for sailing calculations in GIS*, International Hydrographic Review.

Pallikaris, A., Tsoulos, L., Paradissis, D. 2009b. *New calculations algorithms for GIS navigational systmes and receivers*, Proceeidngs of the European Navigational Conference ENC - GNSS, Naples, Italy.

Pallikaris, A., Tsoulos L., Paradissis D. 2010. *Improved algorithms for sailing calculations.* Coordinates, Vol. VI, Issue 5, May, p. 15-18.

Pallikaris, A., Latsas, G. 2009. New algorithm for great elliptic sailing (GES), The Journal of Navigation, vol. 62, p. 493-507.

Prince, R., Williams, R. 1995. *Sailing in ever-decreasing circles*, The Journal of Navigation, Vol. 48, No. 2, p. 307-313.

Rainsford, H.F. 1953. *Long Geodesics on the Ellipsoid*, Bulletin Geodesique, Vol. 30, p. 369-392.

Rainsford, H.F. 1955. *Long Geodesics on the Ellipsoid*, Bulletin Geodesique, Vol. 37.

Sadler, D.H. 1956. *Spheroidal Sailing and the Middle Latitude*, The Journal of Navigation, Vol. 9, issue 4, p. 371-377.

Schechter, M. 2007. *Which Way Is Jerusalem? Navigating on a Spheroid*. The College Mathematics Journal, Vol. 38, No. 2, March.

Silverman, R. 2002. *Modern Calculus and Analytic Geometry*. Courier Dover Publications, Dover.

Sinnott, R. 1984. *Virtues of the Haversine*. Sky and Telescope. Vol. 68, No. 2.

Snyder, J.P. 1987. *Map Projections: A Working Manual*. U.S. Geological Survey Professional Paper 1395.

Sodano, E.M. 1965. *General Non-iterative Solution of the Inverse and Direct Geodetic Problems*, Bulletin Geodesique, Vol. 75.

Tobler, W. 1964. *A comparison of spherical and ellipsoidal measures*, The Professional Geographer, Vol. XVI, No 4, p. 9-12.

Torge, W. 2001. *Geodesy*. 3rd edition. Walter de Gruyter, Berlin, New York.

Tseng, W.K., Lee, H.S. 2007. *The vector function of traveling distance for great circle navigation*, The Journal of Navigation, Vol. 60, p. 158-164.

Tseng, W.K., Lee, H.S. 2010. *Navigation on a Great Ellipse*, Journal of Marine Science and Technology, Vol. 18, No. 3, p. 369-375.

Tyrrell, A.J.R. 1955. *Navigating on the Spheroid*, The Journal of Navigation, vol. 8, 366 (Forum).

Vincenty, T. 1975. *Direct and Inverse Solutions of Geodesics on the Ellipsoid with Application of Nested Equations.* Survey Review, Vol. XXII, No. 176, p. 88-93.

Vincenty, T. 1976. *Direct and Inverse Solutions of Geodesics on the Ellipsoid with Application of Nested Equations – additional formulas.* Survey Review Vol. XXIII, No. 180, p. 294.

Walwyn, P.R. 1999. *The Great ellipse solution for distances and headings to steer between waypoints*, The Journal of Navigation, Vol. 52, p. 421-424.

Weintrit, A. 2009. *The Electronic Chart Display and Information System (ECDIS). An Operational Handbook*. A Balke-ma Book. CRC Press, Taylor & Francis Group, Boca Raton – London - New York - Leiden, pp. 1101.

Weintrit, A., Kopacz, P. 2004. *Safety Contours on Electronic Navigational Charts*. 5th International Symposium 'Information on Ships' ISIS 2004, organized by German Institute of Navigation and German Society for Maritime Technology, Hamburg.

Weisstein, E. *Great Circle*, MathWorld - A Wolfram Web Resource, http://mathworld.wolfram.com/GreatCircle.html;

Willard, S. 1970. *General Topology*, Addison-Wesley. Reprinted by Dover Publications, New York, 2004 (Dover edition).

Williams, J.E.D. 1950. *Loxodromic Distances on the Terrestrial Spheroid*, The Journal of Navigation, Vol. 3, No. 2, p. 133-140.

Williams, E. 2002. *Navigation on the spheroidal Earth*, http://williams.best.vwh.net/ellipsoid/ellipsoid.html.

Williams, R. 1996. *The Great Ellipse on the surface of the spheroid*, The Journal of Navigation, Vol. 49, No. 2, p. 229-234.

Williams, R. 1998. *Geometry of Navigation*, Horwood series in mathematics and applications, Horwood Publishing, Chichester,

Williams, R., Phythian, J.E. 1989. *Navigating Along Geodesic Paths on the Surface of a Spheroid,* The Journal of Navigation, Vol. 42, p. 129-136.

Williams, R., Phythian, J.E. 1992. *The shortest distance between two nearly antipodean points on the surface of a spheroid,* The Journal of Navigation, Vol. 45, p. 114.

Wippern, K.C.C. 1988. *Surface Navigation and Geodesy, A Parametric Approach,* ASFSPA-CECOM Technical Note, March, with addendums;

Wylie, C.R., Barrett, L.C. 1982. *Advanced Engineering Mathematics,*McGraw-Hill, p. 834.

Zukas, T. 1994. *Shortest Spheroidal Distance*, The Journal of Navigation, Vol. 47, p. 107-108.